Wick theorem

- One-point correlator in free theory:

$$\langle \phi(x_1)\rangle_0 = \frac{1}{2}\int d^{d+1}x''\,\Delta(x_1-x'')J(x'') + \frac{1}{2}\int d^{d+1}x'\,\Delta(x'-x_1)J(x')\Big|_{J=0} = 0$$

- Two-point correlator

$$\langle \phi(x_1)\,\phi(x_2)\rangle_0 = \frac{1}{2}\Delta(x_1-x_2) + \frac{1}{2}\Delta(x_2-x_1) = \Delta(x_1-x_2)$$

- Higher-point functions:

 - if N is odd : $\langle \phi(x_1)\dots\phi(x_N)\rangle = 0$

 based on the symmetry of the Lagrangian $\phi \leftrightarrow -\phi$

 - if N is even :

$$N=4: \quad \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_0 = \Delta(x_1-x_2)\,\Delta(x_3-x_4) + \Delta(x_1-x_3)\,\Delta(x_2-x_4) + \Delta(x_1-x_4)\,\Delta(x_2-x_3)$$

General even N:

$$\langle \phi(x_1)\dots\phi(x_N)\rangle_0 = \sum \Delta(x_{j_1}-x_{j_1'})\dots\Delta(x_{j_{N/2}}-x_{j_{N/2}'})$$

sum runs over all distinct ways of grouping the set $\{1,\dots,N\}$ in pairs

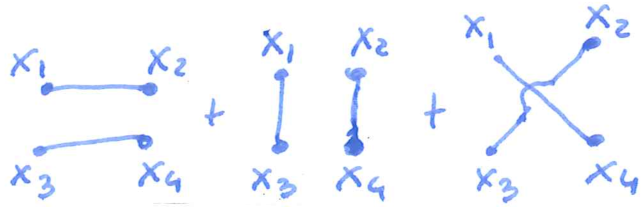
Wick theorem

- Graphical notation:

$= \Delta(x_1-x_2)$

Example: $N=4$

$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle_0 =$



Feynman diagrams

- For interacting theories the Lagrangian splits

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

$\uparrow$ only quadratic in fields

- Correlation functions

$$G^{(N)}(y_1, \dots, y_N) = \frac{\int \mathcal{D}\phi \; \phi(y_1) \dots \phi(y_N)\, e^{-S_0 - \int \mathcal{L}_I(x)\, d^Dx}}{\int \mathcal{D}\phi \; e^{-S_0 - \int \mathcal{L}_I(x)\, d^Dx}}$$

We expand the numerator and denominator in powers of $\mathcal{L}_I$:

$$e^{-\int \mathcal{L}_I(x) d^Dx} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d^Dx_1 \dots d^Dx_n \bullet \mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_n)$$

All terms in the numerator are then of the form

$$\langle \phi(y_1) \dots \phi(y_N)\, \mathcal{L}_I(x_1) \dots \mathcal{L}_I(x_n) \rangle_0$$

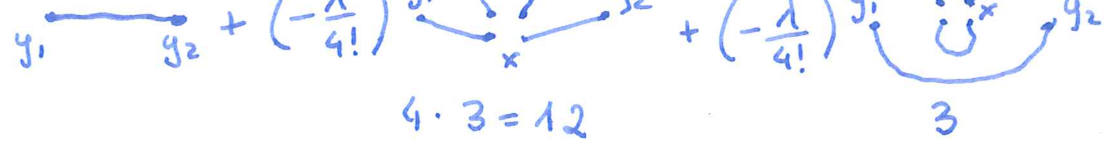
$\rightarrow$ these are correlators in free theory

Since $\mathcal{L}_I$ is a polynomial in ϕ then it can be done by Wick's theorem.

- Example: $\phi^4 \longrightarrow \mathcal{L}_I(x) = \frac{\lambda}{4!}\phi(x)^4$

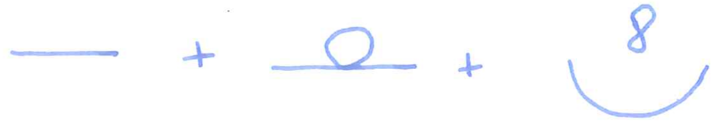
Let us move fields at x a bit apart $\phi(x)^4 \rightarrow \prod_{j=1}^{4} \phi(x_j)$

$G^{(2)}(y_1, y_2)_{NUM} =$



$4 \cdot 3 = 12$ $\qquad$ 3

$$= \Delta(y_1 - y_2) - 12\frac{\lambda}{4!}\int d^Dx\, \Delta(y_1 - x)\Delta(x-x)\Delta(x-y_2) - 3\frac{\lambda}{4!}\Delta(y_1-y_2)\int d^Dx\, \Delta(x-x)^2$$

$G^{(2)}(y_1,y_2)_{NUM} =$ 

Now, the denominator:

$$G^{(2)}(y_1,y_2)_{DEN} = 1 + 8$$
$$= 1 - \frac{3\lambda}{4!}\int \Delta(x-x)^2\, d^Dx$$

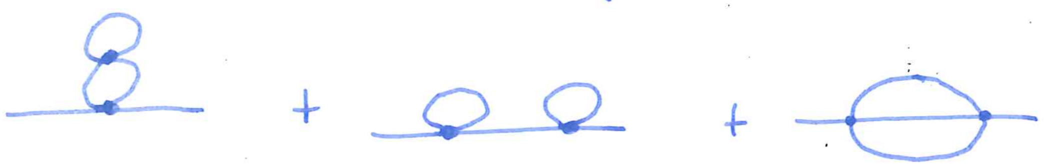
- The integral $\int \Delta(x-x)^2 d^Dx = \Delta^2(0)\int d^Dx$ diverges

$$\int d^Dx = V\cdot T \to \infty$$

This contribution is cancelled by the denominator. We keep $V\cdot T$ finite, divide and only then take $V\cdot T$ to infinity.

→ diagrams containing pieces not connected to the external points (bubbles) can be ignored

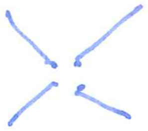
- Second-order two-point function

λ^2: 

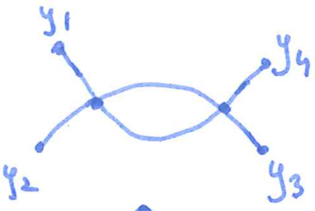
The last one:

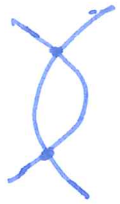
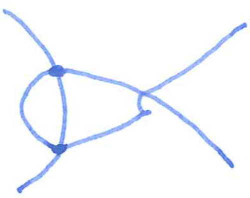
$$\underset{x_1\leftrightarrow x_2}{\underset{\uparrow}{2}}\cdot \underset{x\text{ with }y}{\underset{\uparrow}{4\cdot 4}}\cdot \underset{x_1\text{ with }x_2}{\underset{\uparrow}{3!}}\cdot \frac{1}{2!}\left(-\frac{\lambda}{4!}\right)^2\int \Delta(y_1-x_1)\Delta(x_1-x_2)^3\Delta(x_2-y_2)\,d^Dx_1\,d^Dx_2$$

- Four-point correlation function (only connected contributions)

λ:  = 

$$= 4! \, \frac{-\lambda}{4!} \int \prod_{j=1}^{4} \Delta(y_j - x) \, d^D x$$

λ^2: 

\+  + 

$$\frac{(-\lambda)^2}{2} \int \Delta(y_1 - x_1)\, \Delta(y_2 - x_1)\, \Delta(x_1 - x_2)^2\, \Delta(x_2 - y_3)\, \Delta(x_2 - y_4)\, d^D x_1\, d^D x_2$$

\+  $\leftarrow$ not relevant later

Feynman rules in position space

1. Draw all topologically distinct connected diagrams with N external lines and each internal vertex attached to 4 lines.
2. To each line associate a factor $\Delta(x'-x'')$
3. To each internal vertex associate a factor $-\lambda$
4. Integrate over internal vertices $\prod_j \int d^D x_j$
5. Multiply by the symmetry factor $1/\text{integer}$

- Only the last point might be problematic.
  - → generic diagram symmetry factor is 1.
  - → non-generic diagrams:

 $\longrightarrow \frac{1}{2}$ (twist the bubble)

 $\longrightarrow \left(\frac{1}{2!}\right)^2$

 $\longrightarrow \frac{1}{3!}$ (permutation of 3 internal lines)

In general: symmetry factor is the inverse of number of elements in the symmetry group of a given diagram.

General formula for symmetry factors in ϕ^4

Define:

S – number of self-connections

D – number of double connections

T – number of triple connections

Q – number of quadruple connections

N_{IVP} – number of identical vertex permutations

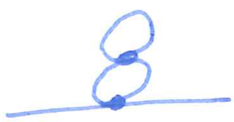
Then:

$$S_G = \frac{1}{2^{S+D}(3!)^T(4!)^Q N_{IVP}}$$

Example:

: $S=1$, $D=0$, $T=0$, $N_{IVP}=1$

$\Rightarrow S_G = \frac{1}{2}$



: $S=1$, $D=1$, $T=0$, $N_{IVP}=1$

$\Rightarrow S_G = \frac{1}{4}$

: $S=0$, $D=0$, $T=1$, $N_{IVP}=1$

$\Rightarrow S_G = \frac{1}{3!}$