Counterterms

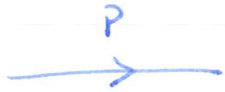
- One can implement these conditions at the level of Lagrangian:

$$\phi_0 = \underbrace{(1+\delta Z_\phi)}_{Z}^{1/2} \phi \ , \quad Z m_0^2 = m^2 + \delta m^2 \ , \quad Z^2 \lambda_0 = \lambda + \delta\lambda$$

Then the Lagrangian density turns into

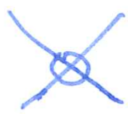
$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 +$$
$$\underbrace{+ \frac{\delta Z_\phi}{2} \partial_\mu \phi \partial^\mu \phi + \frac{\delta m^2}{2}\phi^2 + \frac{\delta \lambda}{4!}\phi^4}_{\text{counterterms}}$$

We end up with a new Lagrangian for which we can derive new Feynman rules:

 $= \dfrac{1}{p^2+m^2}$ ← physical propagator

 $= -\lambda$ ← physical coupling

 $= -p^2 \delta Z_\phi - \delta m^2$ ← correction to the propagator

 $= -\delta\lambda$ ← correction to the coupling

Example:

$\tilde{G}^{(2)} =$ $+$  $+$  $+ \mathcal{O}(\lambda^2)$

$\tilde{G}^{(4)} =$  $+ \Big($  $+$ permutations$\Big) +$  $+ \mathcal{O}(\lambda^2)$

Contributions to $\tilde{G}^{(2)}$:

$$\frac{1}{p^2+m^2} + \left(\frac{-\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2+m^2} - p^2 \delta Z_\phi - \delta m^2 \right) \frac{1}{(p^2+m^2)^2}$$

$$= \frac{1}{p^2+m^2} + \left(-\frac{\lambda}{2} \frac{\pi^{D/2}}{(2\pi)^D} \Gamma(1-D/2)\, m^{D-2} - p^2 \delta Z_\phi - \delta m^2 \right) \frac{1}{(p^2+m^2)^2}$$

Now the condition

$$\Gamma^{(2)}(p^2 = -m^2) = 0$$

implies

$$-\frac{\lambda}{2} \frac{\pi^{D/2}}{(2\pi)^D} \Gamma(1-D/2)\, m^{D-2} - p^2 \delta Z_\phi - \delta m^2$$

$$\Rightarrow \quad \delta Z_\phi = 0 \quad \text{and} \quad \delta m^2 = -\frac{\lambda}{2} \frac{\pi^{D/2}}{(2\pi)^D} \Gamma(1-D/2)\, m^{D-2}$$

Summary

- Write down all divergent diagrams
- Regulate them by splitting off their infinite parts which then are cancelled by counterterms in the Lagrangian
- This procedure is unique after renormalization conditions have been imposed

Renormalization schemes

- Mass-shell renormalization

$$\Gamma_0^{(2)}(p^2=-m^2)=0 \quad , \quad \left.\frac{\partial \Gamma_0^{(2)}}{\partial p^2}\right|_{p^2=-m^2} = Z_\phi^{-1}$$

$$\lambda = \Gamma^{(4)}\left((m,\vec{0}),(m,\vec{0}),(-m,\vec{0}),(-m,\vec{0})\right)$$

We may choose other schemes which are equally valid (and often easier to compute)

- Zero-momentum renormalization

$$m^2 = \Gamma^{(2)}(p=0) \quad , \quad \left.\frac{\partial \Gamma^{(2)}(p)}{\partial p^2}\right|_{p=0} = 1$$

$$\lambda = -\Gamma^{(4)}(p_1=p_2=p_3=p_4=0)$$

Example: renormalized 4-point function in $D=4$

$$\Gamma^{(4)}(p_1,\dots,p_4) = Z_\phi^2\Big(-\lambda_0 + \tfrac{1}{2}\lambda_0^2\left[I_4(p_1+p_2)+I_4(p_1+p_3)+I_4(p_1+p_4)\right] + \mathcal{O}(\lambda_0^3)\Big)$$

where we have already computed:

$$I_4(p) = \frac{\pi^{D/2}}{(2\pi)^D}\,\Gamma(2-D/2)\int_0^1 \left[x(1-x)p^2+m^2\right]^{D/2-2}dx$$

We set $Z_\phi = 1+\mathcal{O}(\lambda_0^2)$ and find

$$\lambda = \lambda_0 - \frac{3}{2}\lambda_0^2\frac{\pi^{D/2}}{(2\pi)^D}\Gamma(2-D/2)\,m^{D-4} + \mathcal{O}(\lambda_0^3)$$

Solve for λ_0:

$$\lambda_0 = \lambda + \frac{3}{2}\lambda^2\frac{\pi^{D/2}}{(2\pi)^D}\Gamma(2-D/2)\,m^{D-4} + \mathcal{O}(\lambda^3)$$

Inserting into $\Gamma^{(4)}$

$$\Gamma^{(4)}(p_1,\dots,p_4) = -\lambda + \frac{1}{2}\lambda^2 \frac{\pi^{D/2}}{(2\pi)^D}\Gamma(2-D/2) \times$$
$$\times \left(\int_0^1 \left[\left(x(1-x)(p_1+p_2)^2 + m^2 \right)^{D/2-2} - m^{D-4} \right] dx + \text{perms} \right)$$

Removing the regulator $D \to 4$

$$\Gamma^{(4)}(p_1,\dots,p_4) = -\lambda - \frac{\lambda^2}{32\pi^2}\left(\int_0^1 \log \frac{x(1-x)(p_1+p_2)^2 + m^2}{m^2} dx + \text{perms} \right)$$

The renormalized correlation functions are finite when expressed in terms of the renormalized parameters.

- Minimal subtraction scheme

We can write in general:

$$\lambda = Z_\lambda^{-1}\lambda_0 \quad , \quad Z_\lambda = 1 + \mathcal{O}(\lambda)$$

contains a pole as $D \to 4$

This contribution cancels the pole of $\Gamma^{(4)}$ at all values of the momenta

More precisely:

$$Z_\lambda = 1 - \frac{\lambda_0}{m^{2\epsilon}}\left(\frac{1}{32\pi^2\epsilon} + \dots \right) + \mathcal{O}(\lambda_0^2)$$

finite as $\epsilon \to 0$

where $D = 4 - 2\epsilon$

In the minimal subtraction scheme we define $\lambda = Z_\lambda^{-1}\lambda_0$ such that Z_λ contains only the pole term.

Massless case and IR divergences

Neither of the schemes described previously work in the case where the renormalized mass $m=0$ (because of the infrared divergences)

Example: 1-loop coupling constant correction

$$\sim \int \frac{d^D p}{p^4} : \begin{array}{l} \text{UV divergent for } D \geq 4 \\ \text{IR divergent for } D \leq 4 \end{array}$$

In order to renormalize them we need to introduce an extra parameter $\tilde{\mu}$ with dimension of mass.

Renormalization conditions:

$$m^2 = \Gamma^{(2)}(p=0) = 0 \quad ; \quad \left.\frac{\partial \Gamma^{(2)}(p)}{\partial p^2}\right|_{p^2=\tilde{\mu}^2} = 1$$

$$\lambda = -\Gamma^{(4)}(p_j \sim \tilde{\mu})$$

Statement of renormalizibility for ϕ^4

Starting from the regularized bare theory with parameters m_0 and λ_0, if we perform:

- field renormalization $\phi = Z_\phi^{-1/2} \phi_0$, e.g. $\left.\frac{\partial \Gamma^{(2)}(p)}{\partial p^2}\right|_{p=0} = 1$
- mass renormalization, e.g. $m^2 = \Gamma^{(2)}(p=0)$
- coupling renormalization, e.g. $\lambda = -\Gamma^{(4)}(p_j=0)$

then for $D \leq 4$ all renormalized N-point functions $\tilde{G}^{(N)}(p_1, \dots, p_N)$ have a finite limit as a regulator is removed

A comment on composite operators

Desribed renormalization procedure guarantees the finiteness of correlation functions in position space only if the points $\{y_j\}$ do not coincide.

$$\langle \phi(y_1) \dots \phi(y_N) \rangle = \int \prod_{j=1}^{N} e^{i p_j y_j} d^D p_j \, \tilde{G}^{(N)}(p_1, \dots, p_N) (2\pi)^D \times \delta^D\Big(\sum_j p_j\Big)$$

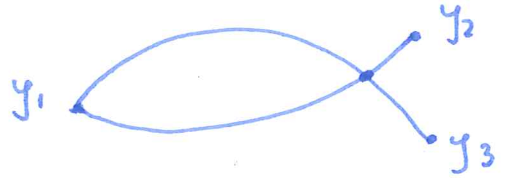
these momentum integrals are damped by phase oscillations of factors $e^{i(y_j - y_{j'})p_j}$

When some of $\{y_j\}$ coincide there are still divergences

Example: one-loop of:

$$\langle \phi_0(y_1)^2 \, \phi_0(y_2) \, \phi_0(y_3) \rangle$$

is given by the diagram



which is logarithmically divergent and it is not made finite by field or coupling renormalization.

This is an example of a composite field.

It requires an additional renormalization:

$$\phi(y)^2 = Z_{\phi^2}^{-1} \phi_0(y)^2$$

Renormalization in other theories

Strategy:

- perform power counting to identify primitively divergent $\Gamma^{(N)}$
- identify the critical dimension D_c for which the coupling constant is dimensionless
- at or just below D_c make finite all divergent diagrams by proper subtractions

Example: $\kappa\phi^6$ theory

$$[\Gamma^{(N)}] = N + D - N\frac{D}{2}$$

and

$$[\kappa] = 6 - 2D \Rightarrow D_c = 3$$

At D_c : $[\Gamma^{(2)}] = 2$, $[\Gamma^{(4)}] = 1$ and $[\Gamma^{(6)}] = 0$

This means that the theory requires:

- mass renormalization
- field renormalization
- coupling renormalization
- counterterm $\sim \phi^4$ in the Lagrangian (even though it was not there in the bare theory