Renormalization group

So far: we have tried to determine when and how the cancellations of UV divergences in QFT take place

- divergences appear only in the values of a few parameters: the bare mass and coupling constant (or in counterterms in renormalized theory)
- aside from that shifts in parameters, the virtual particles with large momenta have no effect on computations

Wilson's approach to the perturbative QFT:

→ all parameters of a renormalizable QFT can be thought of as scale-dependent quantities
- this scale dependence is described by simple differential equations called renormalization group equations

→ one can study the origin of UV divergences by isolating the dependence of the functional integral on the short-distance degrees of freedom of the field.

Let us study a theory with UV cut-off Λ

$$Z_\Lambda[J] = \int_{|p|<\Lambda} \mathcal{D}\phi_0 \; e^{-S + \int d^Dx\, J(x)\,\phi_0(x)}$$

with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_0)^2 + \frac{1}{2} m_0^2 \phi_0^2 + \frac{\lambda_0}{4!}\phi_0^4$$

and the field ϕ_0 in the momentum space is truncated:

$$\phi_0(p) = \begin{cases} \phi_0(p) & , \text{for } |p| < \Lambda \\ 0 & , \text{for } |p| > \Lambda \end{cases}$$

We study what is the influence of high-momentum modes on the physical predictions of the theory by integrating out high-momentum contributions.

Choose a number $0 < b < 1$ and split:

$$\phi_0 = \phi + \hat{\phi}$$

with

$$\phi(p) = \begin{cases} \phi_0(p) & \text{for } |p| < b\Lambda \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{\phi}(p) = \begin{cases} 0 & \text{for } |p| < b\Lambda \\ \phi_0(p) & \text{for } b\Lambda < |p| < \Lambda \end{cases}$$

Then the generating functional (with $J=0$ for simplicity):

$$Z = \int \mathcal{D}\phi \int \mathcal{D}\hat{\phi} \, \exp\Big[-\int d^Dx \Big(\frac{1}{2}(\partial_\mu\phi + \partial_\mu\hat{\phi})^2 + \frac{1}{2}m_0^2(\phi+\hat{\phi})^2 + \frac{\lambda_0}{4!}(\phi+\hat{\phi})^4\Big)\Big]$$

$$= \int \mathcal{D}\phi \, e^{-\int d^Dx\, \mathcal{L}(x)} \int \mathcal{D}\hat{\phi} \, \exp\Big[-\int d^Dx \Big(\frac{1}{2}(\partial_\mu\hat{\phi})^2 + \frac{1}{2}m_0^2\hat{\phi}^2$$

$$+ \lambda_0\Big(\frac{1}{6}\phi^3\hat{\phi} + \frac{1}{4}\phi^2\hat{\phi}^2 + \frac{1}{6}\phi\hat{\phi}^3 + \frac{1}{4!}\hat{\phi}^4\Big)\Big)\Big]$$

We perform the functional integral over $\hat\phi$ and as the result we get:

$$Z = \int_{|p|<b\Lambda} \mathcal{D}\phi \, \exp\left(-\int d^Dx \, \mathcal{L}_{eff}\right)$$

with

$$\mathcal{L}_{eff} = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m_0^2 \phi^2 + \frac{1}{4!}\lambda_0 \phi^4 + \text{the result of integrating out } \hat\phi$$

The leading-order term involving $\hat\phi$ is:

$$\frac{1}{2}\int_{b\Lambda \leq |p| \leq \Lambda} \frac{d^Dp}{(2\pi)^D} \hat\phi(p)\, p^2 \hat\phi(p)$$

which leads to a propagator (assuming $m_0 \ll \Lambda$).

$$\hat\phi(p)\, \hat\phi(p') = \frac{1}{p^2}(2\pi)^D \delta^D(p+p')\, \Theta(p)$$

with $\Theta(p) = \begin{cases} 1 & \text{if } b\Lambda \leq |p| \leq \Lambda \\ 0 & \text{otherwise} \end{cases}$

All other terms are perturbations and can be evaluated using Wick's theorem, e.g. term $\phi^2\hat\phi^2$:

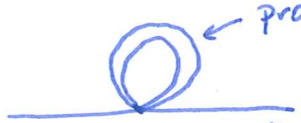
$$-\int d^Dx \, \frac{\lambda_0}{4} \phi^2 \hat\phi \hat\phi = -\frac{1}{2}\int \frac{d^Dp}{(2\pi)^D} \phi(p)\phi(-p)\mu$$

with

$$\mu = \frac{\lambda_0}{(4\pi)^{D/2}\,\Gamma(D/2)} \, \frac{1-b^{D-2}}{D-2} \, \Lambda^{D-2}$$

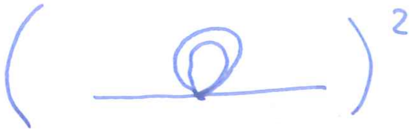
We can adopt a diagrammatic notation:

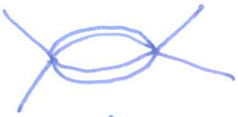
propagator of $\hat\phi$



propagator of ϕ

At order λ_0^2 we have two diagrams:





exponentiate to $e^{\,\cdot}$ 

correction to the four-point function $\sim -\frac{1}{4}\int d^D x \, g \, \phi^4$

with
$$g = -\frac{3\lambda_0^2}{(4\pi)^{D/2}\,\Gamma(D/2)} \frac{1-b^{D-4}}{D-4} \Lambda^{D-4}$$

$$\xrightarrow{D\to 4} -\frac{3\lambda_0^2}{16\pi^2} \log(1/b)$$

In Wilson's treatment this divergence is not pathological $\rightarrow$ it is a sign that the diagram receives contributions from all momentum scales.

- After we integrate out the field $\hat{\phi}$, the Lagrangian $\mathcal{L}(\phi)$ receives corrections. We can now compare the original functional integral with the one we just derived:

Rescale: $p' = \frac{p}{b}$, $x' = x \cdot b$

Schematic form of the effective action

$$\int d^D x \, \mathcal{L}_{eff} = \int d^D x \Big[\frac{1}{2}(1+\Delta Z)(\partial_\mu \phi)^2 + \frac{1}{2}(m_0^2 + \Delta m^2)\phi^2$$
$$+ \frac{1}{4!}(\lambda_0 + \Delta\lambda)\phi^4 + \Delta C (\partial_\mu \phi)^4 + \Delta D \, \phi^6 + \ldots$$

$$\underline{\text{rescale}} \int d^D x' \Big[\frac{1}{2}(1+\Delta Z) b^2 (\partial'_\mu \phi)^2 + \frac{1}{2}(m_0^2 + \Delta m^2)\phi^2$$
$$+ \frac{1}{4!}(\lambda_0 + \Delta\lambda)\phi^4 + \Delta C \, b^4 (\partial'_\mu \phi)^4 + \Delta D \, \phi^6 \Big] b^{-D}$$

Also rescale the field

$$\phi' = \left(b^{2-D}(1+\Delta Z)\right)^{1/2} \phi$$

Finally:

$$\int d^D x \, \mathcal{L}_{eff} = \int d^D x' \left[\frac{1}{2}(\partial'_\mu \phi')^2 + \frac{1}{2} m'^2 \phi'^2 + \frac{\lambda'}{4!} \phi'^4 \right.$$

$$\left. + C'(\partial'_\mu \phi')^4 + D' \phi'^6 + \ldots \right.$$

with

$$m'^2 = (m_0^2 + \Delta m^2)(1+\Delta Z)^{-1} b^{-2}$$

$$\lambda' = (\lambda + \Delta\lambda)(1+\Delta Z)^{-2} b^{-4}$$

$$C' = (C + \Delta C)(1+\Delta Z)^{-2} b^{D}$$

$$D' = (D + \Delta C)(1+\Delta Z)^{-3} b^{2D-6}$$

- By combining the operation of integrating out high-momentum degrees of freedom with rescaling we have rewritten these operations as a transformation of the Lagrangian.
- We could repeat this and integrate out another layer of the momentum space
- If we take the parameter b to be close to 1, the transformation becomes continuous

> Integrating out high-momentum degrees of freedom of a field theory describes a trajectory (flow) in the space of all possible Lagrangians

Now, there are two strategies to compute correlation functions of fields with momenta much smaller than Λ

- use the original Lagrangian $\mathcal{L}$ $\rightarrow$ the effect of high-momentum fluctuations do not show up until we compute loop diagrams $\rightarrow$ large shift from the bare parameters to the values appropriate to low-momentum processes appear suddenly in one-loop diagrams.
- use the effective Lagrangian $\mathcal{L}_{eff}$ $\rightarrow$ the effect of high-momentum fluctuations has been absorbed into new coupling constants $(m', \lambda', \ldots)$, their influence can be seen directly from the Lagrangian.

Both approaches should produce the same result. However, the parameters of the effective Lagrangian may be very different from those of the original theory.

How Lagrangians tend to vary under RG flow?

- The simplest case is the vicinity of the point

$$m^2 = \lambda = C = D = \dots = 0 \qquad (*)$$

→ free-theory Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi)^2$$

It is a <u>fixed-point</u> of the renormalization group.

- In the vicinity of (*) we ignore terms $\Delta m^2, \Delta\lambda, \dots$ and focus on terms linear in perturbations:

$$m'^2 = m^2 b^{-2}, \lambda' = \lambda b^{D-4}, C' = C b^{D}, D' = D b^{2D-6}, \dots$$

with $m^2 \sim 0$, $\lambda \sim 0$, $C \sim 0$, $D \sim 0, \dots$

- Since $b < 1$, those parameters that we multiplied by negative powers of b grow, while those that are multiplied by positive powers of b decay.

- We think of various terms in the effective Lagrangian as a set of local operators that can be added as perturbations to $\mathcal{L}_0$

- We distinguish three-cases:
  - coefficient grows : relevant operator
  - coefficient decays : irrelevant operator
  - multiplied by b^0 : marginal operator

For generic operators with N powers of ϕ and M derivatives the coefficient transforms as:

$$C'_{N,M} = b^{\underbrace{N(D/2-1)+M}_{\text{mass dimension of the operator}} - D} \; C_{N,M}$$

- There is a relation between these three types of operators and renormalizibility:
  - relevant operators $\leftrightarrow$ superrenormalizable
  - marginal operators $\leftrightarrow$ renormalizable
  - irrelevant operators $\leftrightarrow$ non-renormalizable (but truly irrelevant for low-energy physics)