Callan - Symanzik equation

Focus on massless theories in $D = 4-2\epsilon$ dimensions, $\epsilon > 0$.

- Introduce dimensionless renormalized coupling constant

$$g \equiv \lambda \mu^{-2\epsilon} = \sum_{n=0}^{\infty} \alpha_n \left(\lambda_0 \mu^{-2\epsilon}\right)^n$$

We assume that the renormalized mass is

$$m = 0$$

Then the bare mass can be expressed as a function of the bare coupling λ_0 and the renormalization scale μ.

- The statement of renormalizibility:

$$\Gamma^{(N)}(\{p_j\}, g, \mu) = Z_\phi^{N/2}(\lambda_0, \mu)\, \Gamma_0^{(N)}(\{p_j\}, \lambda_0)$$

this has a finite limit when we remove the regulator: $\epsilon \to 0$.

- Bare vertex function does not know anything about the scale μ:

$$\mu \frac{d}{d\mu} \Gamma_0(\{p_j\}, \lambda_0)\Big|_{\lambda_0\text{-fixed}} = 0$$

- Then for the LHS:

$$\mu \frac{d}{d\mu} \left(Z_\phi^{-N/2}(\lambda_0, \mu)\, \Gamma^{(N)}(\{p_j\}, g, \mu)\right)\Big|_{\lambda_0\text{-fixed}} = 0$$

- Using the chain rule:

$$\left(\mu \frac{\partial}{\partial \mu} + \mu \frac{\partial g}{\partial \mu}\Big|_{\lambda_0} \frac{\partial}{\partial g} - \frac{N}{2} Z_\phi^{-1}\, \mu \frac{\partial Z_\phi}{\partial \mu}\Big|_{\lambda_0} \right) \Gamma^{(N)}(\{p_j\}, g, \mu) = 0$$

- We introduce

$$\beta(g) = \mu \frac{\partial g}{\partial \mu}\Big|_{\lambda_0} \quad , \quad \gamma_\phi(g) = \mu \frac{\partial \log Z_\phi}{\partial \mu}\Big|_{\lambda_0}$$

Both functions are independent of μ.

- Finally we arrive to:

$$\boxed{\left(\mu \frac{\partial}{\partial \mu} + \beta(g)\frac{\partial}{\partial g} - \frac{N}{2}\gamma_\phi(g)\right)\Gamma^{(N)}(\{p_i\}, g, \mu) = 0}$$

- In massless theories and from the dimensional analysis we have:

$$\Gamma^{(2)}(p) \sim p^2 F\left(\frac{p^2}{\mu^2}\right)$$

It satisfies the Euler equation:

$$\left(\mu \frac{\partial}{\partial \mu} + p\frac{\partial}{\partial p} - 2\right)\Gamma^{(2)}(p, g, \mu) = 0$$

Subtracting it from the Callan-Symanzik equation

$$\left(p\frac{\partial}{\partial p} - \beta(g)\frac{\partial}{\partial g} - (2 - \gamma_\phi(g))\right)\Gamma^{(2)}(p, g, \mu) = 0$$

- Now, we can trade the dependence on p for the dependence on g. In particular, if we define the <u>running coupling</u> $g(p)$ then

$$\Gamma^{(2)}(p, g, \mu) = \exp\left(\int_g^{g(p)} \frac{2 - \gamma_\phi(g')}{\beta(g')} dg'\right)\Gamma^{(2)}(\mu, g(p), \mu)$$

satisfies the CS eq. if the running coupling satisfies

$$p\frac{\partial}{\partial p} g(p) = \beta(g(p)) \quad , \quad g(\mu) = g$$

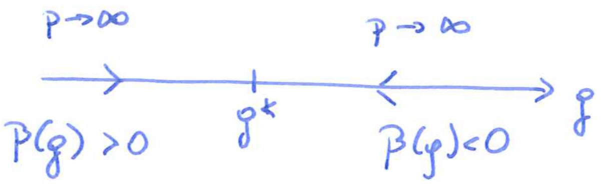
- This is an RG flow equation which defines how the coupling changes along the renormalization group flow. Different renormalizable QFTs have different β-functions and therefore qualitatively different UV and IR behaviours.

- Solution to the RG flow equation:

$$p = \mu \exp\left(\int_{g}^{g(p)} \frac{dg'}{\beta(g')} \right)$$

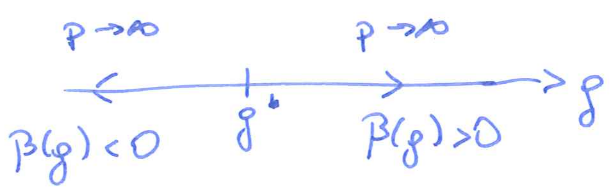
We distinguish three cases:

- $\beta(g) > 0 \Rightarrow g'(p) > 0$ (e.g. $p \nearrow \Leftrightarrow g \nearrow$)
- $\beta(g) < 0 \Rightarrow g'(p) < 0$ (e.g. $p \nearrow \Leftrightarrow g \searrow$)
- if $\beta(g) = 0$ at some point $g = g^*$ then
  - $\rightarrow$ if $\beta(g)$ changes the sign from >0 to <0



then $p \to \infty \Rightarrow g \to g^*$. This is a <u>UV stable point</u>

  - $\rightarrow$ if $\beta(g)$ changes the sign from <0 to >0



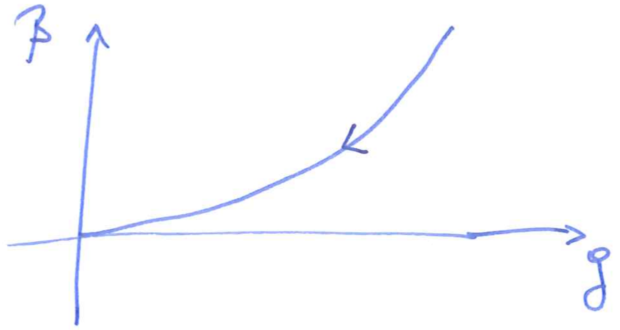
then $p \to 0 \Rightarrow g \to g^*$. This is an <u>IR stable point</u>

Renormalization group flows

Example: ϕ^4

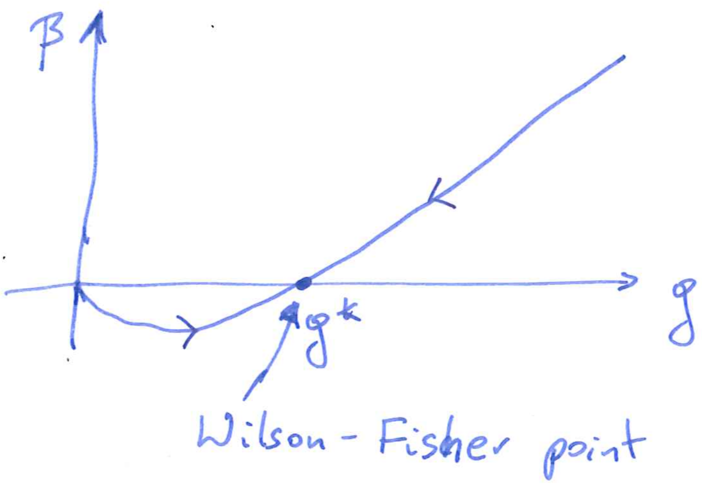
There are two qualitatively different cases:

- $D=4$



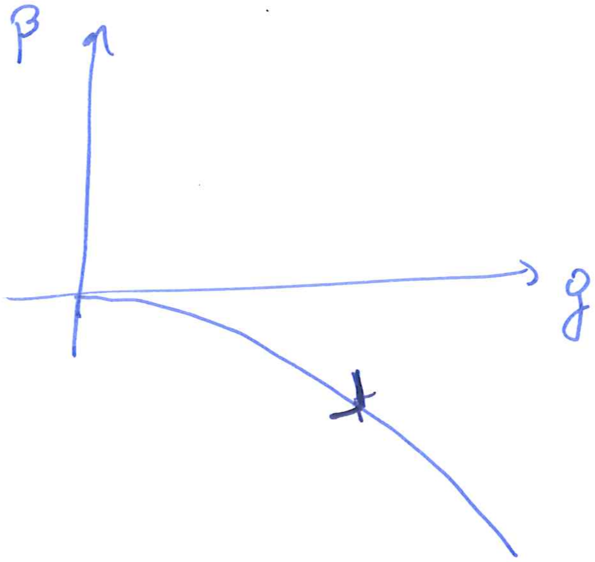
a single IR stable point at $g=0$

- $D<4$



an IR stable point at $g=g^*$ and a UV stable point at $g=0$

Example: QCD



- a UV stable point at $g=0$
  $\rightarrow$ asymptotic freedom
  (the interaction is asymptotically weaker at large energies or small distances)
- as $p \to 0$, $g(p) \to \infty$
  $\rightarrow$ confinement

Anomalous dimension

Assume that the theory has a UV(IR) stable point at $g = g^*$. Leading behaviour of correlators around $g = g^*$:

$$\left(p \frac{\partial}{\partial p} - (2 - \gamma_\phi^*) \right) \Gamma^{(2)}(p) = 0$$

with
$$\gamma_\phi^* = \gamma_\phi(g^*)$$

We can solve it:

$$\Gamma^{(2)}(p) \sim p^{2-\gamma_\phi^*}$$

and get
$$\tilde{G}^{(2)}(p,\mu) \sim \frac{1}{p^2} \left(\frac{p}{\mu} \right)^{\gamma_\phi^*}$$

In position space
$$\langle \phi(x)\, \phi(0) \rangle \sim \frac{1}{|x|^{D-2+\gamma_\phi^*}}$$

Interpretation: The field $\phi(x)$ with canonical dimension $\frac{D-2}{2}$ acquire an anomalous dimension

$$\Delta = \underbrace{\frac{D-2}{2} + \frac{\gamma_\phi^*}{2}}_{\text{scaling dimension}}$$

One-loop computations in ϕ^4 theory

· In the minimal subtraction the coupling constant is:

$$g = \lambda\mu^{-2\epsilon} = \mu^{-2\epsilon}\left(\lambda_0 - \frac{3}{32\pi^2}\frac{1}{\epsilon}\lambda_0^2\mu^{-2\epsilon} + \mathcal{O}(\lambda_0^3)\right)$$

Then:

$$\beta(g) = \mu\frac{\partial g}{\partial\mu}\Big|_{\lambda_0} = -2\epsilon g + \mu^{-2\epsilon}\left(\frac{3}{16\pi^2}\lambda_0^2\mu^{-2\epsilon} + \mathcal{O}(\lambda_0^3)\right)$$

$$= -2\epsilon g + \frac{3}{16\pi^2}(\lambda_0\mu^{-2\epsilon})^2 + \mathcal{O}(\lambda_0^3)$$

$$= -2\epsilon g + \frac{3}{16\pi^2}g^2 + \mathcal{O}(g^3)$$

For $\epsilon = 0$ $(D=4)$ we see that it is positive with a stable point at $g = 0$.

For $\epsilon > 0$, small, there is an IR stable zero at

$$g^* = \frac{2\cdot 16\pi^2}{3}\epsilon + \mathcal{O}(\epsilon^2)$$

• Evaluation of γ_ϕ:

$$Z_\phi^{-1} = \frac{\partial\Gamma_0^{(2)}(p)}{\partial p^2}\Big|_{p^2=\mu^2} = 1 - \frac{\lambda_0^2}{3!}\frac{\partial}{\partial p^2}\int\frac{d^Dk_1 d^Dk_2}{(2\pi)^{2D}}\frac{1}{k_1^2k_2^2(p-k_1-k_2)^2} + \mathcal{O}(\lambda_0^3)$$

$$= 1 - \frac{1}{24\epsilon}\frac{\lambda_0^2\mu^{-4\epsilon}}{(16\pi^2)^2} + \mathcal{O}(\lambda_0^3)$$

so

$$\gamma_\phi = \mu\frac{\partial}{\partial\mu}\log Z_\phi = \frac{1}{6}\left(\frac{g}{16\pi^2}\right)^2 + \mathcal{O}(g^3)$$

At the stable point:

$$\gamma_\phi^* = \frac{2\epsilon^2}{27} + \mathcal{O}(\epsilon^3)$$

Anomalous dimension of $\phi^2(x)$

Define

$$\tilde{G}_0^{(2,1)}(p_1, p_2) = \int d^D y_1 d^D y_2 \, e^{i(p_1 y_1 + p_2 y_2)} \langle \phi_0(0)^2 \phi_0(y_1) \phi_0(y_2) \rangle$$

and a vertex function

$$\Gamma_0^{(2,1)}(p_1, p_2) = \frac{\tilde{G}_0^{(2,1)}(p_1, p_2)}{\tilde{G}_0^{(2)}(p_1)\, G_0^{(2)}(p_2)} \Bigg|_{1PI}$$

Renormalization of ϕ^2:

$$\phi(x)^2 = Z_{\phi^2}^{-1} \phi_0(x)^2$$

so

$$\Gamma^{(2,1)}(p_1, p_2) = Z_{\phi^2}^{-1} Z_\phi \Gamma_0^{(2,1)}(p_1, p_2)$$

The renormalization constant Z_{ϕ^2} is fixed by

$$\Gamma^{(2,1)}(p_1, p_2)\Big|_{(p_1+p_2)^2 = \mu^2} = 1$$

The Callan-Symanzik equation:

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} - \gamma_\phi(g) + \gamma_{\phi^2}(g) \right) \Gamma^{(2,1)}(p_1, p_2, \mu) = 0$$

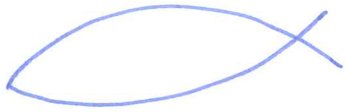
with

$$\gamma_{\phi^2}(g) = \mu \frac{\partial}{\partial \mu} \log Z_{\phi^2} \Big|_{\lambda_0}$$

Then the solution is given by (around $g = g^*$)

$$\Gamma^{(2,1)}(p_1, p_2) \sim p^{\gamma_{\phi^2}^* - \gamma_\phi^*} \quad \text{as } p_1 \sim p_2 \sim p \sim 0$$

Calculate:



$$Z_{\phi^2} = 1 - \frac{1}{32\pi^2}\frac{1}{\epsilon}\lambda_0 \mu^{-2\epsilon} + \mathcal{O}(\lambda_0^2)$$

$$\Rightarrow \gamma^*_{\phi^2} = \mu\frac{\partial}{\partial\mu}\log Z_{\phi^2} = \frac{\lambda_0\mu^{-2\epsilon}}{16\pi^2} + \mathcal{O}(\lambda_0^2) = \frac{g}{16\pi^2} + \mathcal{O}(g^2)$$

At the fixed point $\left(g = \frac{32\pi^2\epsilon}{3}\right)$

$$\gamma^*_{\phi^2} = \frac{2}{3}\epsilon + \mathcal{O}(\epsilon^2)$$