**Quantum Field Theory**

**Homework Assignment #1**

Your tutor will be in touch to tell you the due date for your session;
it will be either the latter part of week 2 or early in week 3

1. This problem combines a review of the harmonic oscillator with that of complex coordinates. In ordinary classical mechanics, consider a two-dimensional harmonic oscillator with Lagrangian
$$L = \frac{1}{m}(\dot{q_1}^2 + \dot{q_2}^2) - \frac{1}{2}m\omega_0^2(q_1^2 + q_2^2),$$
where to get you in the field-theory mood I've labeled $x$ and $y$ as $q_1$ and $q_2$. Now rewrite these coordinates as
$$z \equiv (q_1 + iq_2)/\sqrt{2}\ , \quad z^* \equiv (q_1 - iq_2)/\sqrt{2}\ .$$
(a) Find the classical equations of motion in terms of $q_1$ and $q_2$.
(b) Then rederive them in terms of $z$ and $z^*$, not by just plugging into the preceding, but by rewriting $L$ in terms of $z$ and $z^*$, and then minimising the action under variations $z \to z + \delta z$ and $z^* \to z^* + \delta z^*$. Treat these variations as *independent*.
(c) Find the Hamiltonian in terms of $z$, $z^*$ and the corresponding canonical momenta.
(d) Now write the quantum Hamiltonian using raising and lowering operators $a_1$, $a_2$, $a_1^\dagger$ and $a_2^\dagger$ defined in the usual way. Then rewrite in terms of
$$A \equiv (a_1 + ia_2)/\sqrt{2}\ , \quad B \equiv (a_1 - ia_2)/\sqrt{2}\ .$$
(e) Find the commutations relation for $A$, $A^\dagger$, $B$ and $B^\dagger$ from those for $a_1$ etc., and check that indeed they are an independent pair of rasing/lowering operators.

2. This problem is a review of integration by residue. Evaluate the integrals
$$\int \frac{dp}{2\pi}\frac{e^{ipx}}{p^2 + m^2}$$
$$\int \frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}\cdot\vec{x}}}{p^2 + m^2}$$

3. Show that the expectation value
$$\langle x|e^{-iHt/\hbar}|y\rangle$$
is non-vanishing in quantum mechanics as $|\vec{x} - \vec{y}| \gg ct$, for both non-relativistic and relativistic free-particle Hamiltonians. **Hint:** Use the method of stationary phase.

4. (Peskin, problem 2.1 part(a)): Classical electromagnetism with no sources follows from the action

$$S = -\frac{1}{4}\int dt \int d^3x F_{\mu\nu}F^{\mu\nu} \ , \qquad \text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ ,$$

where I've used standard relativistic 4-vector notation with sums over repeated indices. If you've forgotten it or it's new to you, the four-vector entry in Wikipedia is good. Derive Maxwell's equations as the Euler-Lagrange equations of this action, treating the components $A_\mu(\vec{x}, t)$ as the dynamical variables. Rewrite the equations in terms of electric and magnetic fields by using the definition $E^j = -F^{0j}$ and $\epsilon^{ijk}B^k = -F^{ij}$, where $i, j, k$ are spatial coordinates.

5. Work out the Hamiltonian and equations of motion for a classical field theory with (non-Lorentz-invariant) Lagrangian density

$$\mathcal{L} = \left(\frac{\partial \phi}{\partial t}\right)^2 - \kappa \left(\nabla^2 \phi\right)^2$$

You'll need to slightly generalize the derivation of the Euler-Lagrange equations given in class. The corresponding quantum field theory is often called the "quantum Lifshitz model". Those of you taking topological quantum theory might find it interesting to know that this field theory in two spatial dimensions arises when describing the continuum limit of the square-lattice quantum dimer model, a critical point arising when moving out of a $\mathbb{Z}_2$ topological phase.