**Quantum Field Theory**

**Homework Assignment #4**

due week 8, as arranged by your tutor

1. With the definition that time ordering for fermions obeys

$$T\big(\psi(x)\overline{\psi}(y)\big) \equiv \begin{cases} \psi(x)\overline{\psi}(y) & \text{for } x^0 > y^0 \\ -\overline{\psi}(y)\psi(x) & \text{for } x^0 < y^0 \end{cases}$$

show by explicit calculation that

$$\langle 0|T\big(\psi(x)\overline{\psi}(y)\big)|0\rangle = S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon} \ .$$

You'll need the spin sums

$$\sum_s u^{(s)}(p)\overline{u}^{(s)}(p) = \not{p} + m \ , \qquad \sum_s v^{(s)}(p)\overline{v}^{(s)}(p) = \not{p} - m \ .$$

2. Pions are bosonic particles that occur as bound states of quarks. There are three types, typically labelled as $\pi^+$, $\pi^-$ and $\pi^0$, with the superscript labeling their charge. To avoid confusion with canonical momentum, I will describe $\pi^0$ by a real field $\phi$, and the other two by a complex field $\Phi$, whose particles correspond to $\pi^+$ and antiparticles $\pi^-$. For fairly deep reasons, their interactions among pions must involve derivatives. Their effective Lagrangian density is then

$$\begin{aligned}\mathcal{L}_\pi =& \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) + \frac{1}{2}m^2\phi^2 + (\partial_\mu\Phi^*)(\partial^\mu\Phi) + m^2\phi^2 \\ &+ \frac{1}{6f^2}\Big(\Phi^*\phi(\partial_\mu\Phi)(\partial^\mu\phi) + \Phi\phi(\partial_\mu\Phi)^*(\partial^\mu\phi) - 2\phi\phi(\partial_\mu\Phi^*)(\partial^\mu\Phi)\Big)\end{aligned}$$

(a) Write down the momentum-space Feynman rules for $\mathcal{L}_\pi$, including propagators. Use arrows appropriately to distinguish between particles and antiparticles in $\Phi$, and label momenta as well.

(b) Find the cross-section for the process $\pi^0\pi^0 \to \pi^+\pi^-$ at lowest non-trivial order. You do not need to do the momentum space integrals.

3. Now couple the pions to a proton $P$ and a neutron $N$ via the modified Yukawa couplings

$$\mathcal{L}_Y = g\big(\overline{P}\gamma^5 P\phi - \overline{N}\gamma^5 N\phi + \sqrt{2}\,\overline{P}\gamma^5 N\Phi + \sqrt{2}\,\overline{N}\gamma^5 P\Phi^*\big)$$

so that

$$\mathcal{L} = \mathcal{L}_\pi + \mathcal{L}_P + \mathcal{L}_N + \mathcal{L}_Y$$

where $\mathcal{L}_P$ and $\mathcal{L}_N$ are the usual Dirac Lagrangian densities for the proton and neutron.

(a) Write down the additional Feynman rules applicable to this Lagrangian, making sure you distinguish between which lines are proton lines and which are for neutrons, and which are for particles and which are for anti-particles.

(b) Find the scattering amplitude for $P\pi^0 \to N\pi^+$ at lowest non-trivial order. Your expression may have the $u, v, \overline{u}, \overline{v}$ and/or $\gamma$ in it; you do not need to evaluate them any further.

4. (adapted from Peskin problem 3.4) As I stated in class, there is a way to give a mass to a two-component fermion field that transforms under the Lorentz group as $\psi_L$, the upper two components of the Dirac field. Call this two-component field $\xi$. Consider the following equation for $\chi$ analogous to the Dirac equation:

$$i\overline{\sigma}\cdot\partial\chi - m\sigma^y\chi^* = 0 \tag{1}$$

where $\sigma^\mu$ is the 4-vector $(1,\vec{\sigma})$.

(a) Show that the two terms transform in the same way under the Lorentz group, and so the equation is properly Lorentz covariant. Hint: for the first piece, you can use the fact that $\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda_\mu^\nu\gamma^\nu$, and work in the Weyl/chiral basis.

(b) Show that this equation (1) and its conjugate imply that $\chi$ also satisfies the Klein-Gordon equation $(\partial^2 + m^2)\chi = 0$.

(c) This equation (1) looks great, but it's not immediately apparent how to get it as an equation of motion from a classical action. You might think the mass would come from a term $m(\sigma^y)_{ab}\chi_a^*\chi_b^*$ in the Lagrangian density, but the problem is then that $\sigma^y$ is *antisymmetric*. Thus if $\chi_a^*\chi_b^* = \chi_b^*\chi_a^*$, such a term vanishes. The solution to this issue is to use *Grassmann variables*. Two Grassmann variables $\alpha$ and $\beta$ are defined so that $\alpha\beta = -\beta\alpha$, so that in particular $\alpha^2 = 0$ for any Grassmann variable. The other key definition is that when taking a complex conjugate of a product of Grassmann variables, you reverse the order:

$$(\alpha\beta)^* = \beta^*\alpha^* = -\alpha^*\beta^* \ ,$$

just like you do when taking the hermitian conjugate of matrices. Then a Grassmann field is defined as

$$\xi(x) = \sum_n \alpha_n\phi_n(x)$$

where the $\alpha_n$ are Grassmann variables and the $\phi_n(x)$ are ordinary complex-number functions. So now consider the action

$$S = \int d^4x\Big(\chi^\dagger i\overline{\sigma}\cdot\partial\chi + i\frac{m}{2}\left(\chi^T\sigma^y\chi - \chi^\dagger\sigma^y\chi^*\right)\Big) \ ,$$

where the two components in $\chi$ are Grassmann functions, and $\chi^\dagger = (\chi^*)^T$. Show that $S$ is real, and that varying it with respect to $\chi$ and $\chi^*$ yields the Majorana equation.

**Some uses of Grassman variables:** 1) Path integrals for fermions require them; 2) Peskin problem 3.5 describes how supersymmetry works, using a classical action with Grassmann variables; 3) an action with Grassmann variables provides the easiest way to compute the partition function of two-dimensional dimer models on any planar graph.

5. In the tight-binding approximation, the electrons in graphene hop between nearest-neighbour sites of a honeycomb lattice. There are two sublattices $A$ and $B$, each comprised of half the sites such that all nearest neighbours of $A$ sites are $B$ sites. Thus the hopping Hamiltonian is

$$H = -t\sum_{\langle jj'\rangle}\left(c_{B,j'}^\dagger c_{A,j} + c_{A,j}^\dagger c_{B,j'}\right) \tag{2}$$

where the $\langle jj'\rangle$ is shorthand for summing over $j \in A$ and $j' \in B$ such that $j$ and $j'$ are nearest neighbours. *Two possibly interesting pieces of information:* Graphene is really two copies of the above, one for spin-up electrons, one for spin down. Moreover, graphene is at half-filling,

giving a symmetry between particles and holes. Thus the Fermi "surface" here is thus at $E = 0$ and there is no diagonal term in the Hamiltonian.

The purpose of this problem is diagonalise the Hamiltonian (2 by doing a discrete Fourier transformation. Thus consider the plane-wave states

$$\psi_A(\vec{k}) \equiv \sum_{j\in A} e^{i\vec{k}\cdot\vec{x}_j} c^\dagger_{A,j}|0\rangle \ , \qquad \psi_B(\vec{k}) \equiv \sum_{j'\in B} e^{i\vec{k}\cdot\vec{x}_{j'}} c^\dagger_{B,j'}|0\rangle \ , \tag{3}$$

where $x_j$ is the location of the $j$th site. You won't need to worry about the normalisation of these states.

(a) Find precisely how the Hamiltonian acts on the states in (3). Namely, show that it is block diagonal for each $\vec{k}$, and for each such block,

$$H(\vec{k}) = -t \begin{pmatrix} 0 & f(\vec{k}) \\ f^*(\vec{k}) & 0 \end{pmatrix}$$

with

$$f(\vec{k}) \equiv \sum_{b=1,2,3} e^{i\vec{k}\cdot\vec{\nu}_b} \ ,$$

where the $\nu_b$ are the displacements between an $A$ site and its three nearest neighbours:

$$\vec{\nu_1} = a(0, \frac{1}{\sqrt{3}}) \ , \quad \vec{\nu_2} = a(-\frac{1}{2}, -\frac{1}{2\sqrt{3}}) \ , \quad \vec{\nu_3} = a(\frac{1}{2}, -\frac{1}{2\sqrt{3}}) \ .$$

I have normalized the displacements so that the hexagonal lattice is invariant under shifts by $a(\frac{1}{2}, \frac{\sqrt{3}}{2})$, of magnitude $a$.

(b) Find the energy eigenvalues $E(\vec{k})$ in terms of $|f(\vec{k})|$, and give their explicit values when $\vec{k} = 0$. (These turn out to be the minimum and maximum energies possible).

(c) Show that for

$$\vec{k}_+ \equiv \left(\frac{4\pi}{3a}, 0\right) \ , \qquad \vec{k}_- = -\vec{k}_+ \ ,$$

$E(\vec{k_\pm}) = 0$. (There are other values as well, but these turn out to be equivalent to these two because of the periodicity of $\vec{k}$-space.)

(d) Consider $\vec{k} = \vec{k_+} + \vec{K}$ for $\vec{K}$ small in magnitude. Expand $H(\vec{k})$ to first order in $\vec{K}$, and so show that to this order

$$H(\vec{k}) \propto \sigma^x K^x + \sigma^y K^y \ . \tag{4}$$

(e) In 2+1 spacetime dimensions, we can take the $\gamma$ matrices to be $\gamma^0 = \sigma^z$, $\gamma^1 = i\sigma^x$ and $\gamma^2 = i\sigma^y$. Using these, show that the Hamiltonian for a massless Dirac fermion in 2+1d takes the same form as (4).