Supersymmetry & Supergravity: Problem sheet 1

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Due by Thursday, week 2 (January 24th), 5pm. Problem 6 is optional.

1. On the Poincaré algebra.

Consider the $so(p,q)$ "Lorentz" algebra, leaving invariant the flat metric $\eta_{\mu\nu}$ of $\mathbb{R}^{p,q}$. It is given by:

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}\right) \ . \tag{0.1}$$

(1.a) Check explicitly that $M_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu]$ satisfy the $so(p,q)$ algebra, provided the gamma matrices satisfy the Clifford algebra:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \ . \tag{0.2}$$

(1.b) Write down the matrices ${(M_{\mu\nu})_\rho}^\sigma$ for the fundamental representation of $so(p,q)$ (that is, acting on covectors X_μ).

(1.c) The Poincaré algebra (in any signature) contains the Lorentz generators $M_{\mu\nu}$ and the translation generators P_μ, with the commutation relations:

$$[P_\mu, P_\nu] = 0 \ , \qquad [M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \ . \tag{0.3}$$

Given that:

$$P_\mu = -i\partial_\mu \ , \tag{0.4}$$

on scalar fields $\phi(x)$ (*i.e.* functions of x^μ), what is the expression for $M_{\mu\nu}$ acting on $\phi(x)$? Check that the Poincaré algebra is satisfied. (For instance, $[P_\mu, P_\nu]\phi(x) = -(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\phi(x) = 0$.)

2. Useful identities.

(2.a) Prove the following identities for the σ matrices defined in the lectures:

$$\begin{aligned}
&\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\dot\beta\beta}_\mu = -2\delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha} \ , \\
&\mathrm{Tr}(\sigma^\mu\bar\sigma^\nu) = -2\eta^{\mu\nu} \ , \\
&{(\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu)_\alpha}^\beta = -2\eta^{\mu\nu}\delta^\beta_\alpha \ .
\end{aligned} \tag{0.5}$$

Using these, invert the map:

$$\widetilde{X}_{\alpha\dot{\alpha}} = X_\mu \sigma^\mu_{\alpha\dot{\alpha}} \tag{0.6}$$

from vector to bi-spinor.

(2.b) Check also:

$$\mathrm{Tr}(\sigma^{\mu\nu}\sigma^{\rho\sigma}) = -\frac{1}{2}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - \frac{i}{2}\epsilon^{\mu\nu\rho\sigma} \ , \tag{0.7}$$

with $\epsilon^{0123} = 1$.

3. On $SO(1,3)$.

(3.a) Show that the $so(1,3)$ algebra (0.1) can be decomposed as:

$$so(1,3) \cong su(2) \times su(2)^* \ . \tag{0.8}$$

Hint: first, write down the algebra in terms of the $SO(3)$ rotation and boost generators:

$$J^i = \frac{1}{2}\epsilon^{ijk}M_{jk} \ , \qquad K_i = M_{0i} \ . \tag{0.9}$$

Then, show that:

$$J_i^\pm = \frac{1}{2}(J_i \pm iK_i) \tag{0.10}$$

generate the two $su(2)$ factors.

(3.b) We briefly mentioned in the lectures that there is a *group homomorphism*:

$$Sl(2,\mathbb{C}) \to SO(1,3) \tag{0.11}$$

Given a four-vector X_μ, we have a map to a 2×2 complex matrix, using the σ-matrices:

$$\widetilde{X} = X_\mu \sigma^\mu \ . \tag{0.12}$$

On X_μ, we have the action of an $SO(1,3)$ matrix Λ:

$$X_\mu \to \Lambda_\mu{}^\nu X_\nu \ , \tag{0.13}$$

while on the matrix, the corresponding action is:

$$\widetilde{X} \to N\widetilde{X}N^\dagger \ . \tag{0.14}$$

Explain why N should be an $SL(2,\mathbb{C})$ matrix. Work out the explicit map from N to Λ, and check that it is an homomorphism. Is it one-to-one?

(3.c) Write down explicitly the $SO(1,3)$ matrices acting on left- and right-handed Weyl spinors.

4. On 4d Weyl spinors and Fierzing.

(4.a) Prove that, for Grassmann-number-valued Weyl spinors:

$$\begin{aligned} \psi\chi &= \chi\psi \ , \\ \psi\sigma^\mu\bar\chi &= -\bar\chi\bar\sigma^\mu\psi \ , \end{aligned} \tag{0.15}$$

using our Wess-and-Bagger conventions. Check that $\psi\chi$ is a scalar under $SL(2,\mathbb{C})$. (Given the action, $\psi_\alpha \to N_\alpha{}^\beta\psi_\beta$ on ψ_α, one should first determine the action of $SL(2,\mathbb{C})$ on $\psi^\alpha \equiv \epsilon^{\alpha\beta}\psi_\beta$.)

(4.b) For $\psi, \theta, \chi, \cdots$ some Grassmann-valued spinors, check the Fierz identities:

$$\begin{aligned} (\theta\psi)(\bar\chi\bar\eta) &= \frac{1}{2}(\theta\sigma^\mu\bar\eta)(\bar\chi\bar\sigma_\mu\psi) \ , \\ (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) &= -\frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar\theta\bar\theta) \ . \end{aligned} \tag{0.16}$$

5. The 4d $\mathcal{N}=1$ super-Poincaré algebra.

The 4d $\mathcal{N}=1$ supersymmetry algebra reads:

$$\begin{aligned} &\{Q_\alpha, \bar Q_{\dot\beta}\} = 2\sigma^\mu_{\alpha\dot\beta}P_\mu \ , \\ &\{Q_\alpha, Q_\beta\} = \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0 \ , \\ &[P_\mu, Q_\alpha] = [P_\mu, \bar Q_{\dot\alpha}] = 0 \ , \\ &[M_{\mu\nu}, Q_\alpha] = i(\sigma_{\mu\nu}Q)_\alpha \ , \\ &[M_{\mu\nu}, \bar Q_{\dot\alpha}] = -i(\bar Q\bar\sigma_{\mu\nu})_{\dot\alpha} \ , \end{aligned} \tag{0.17}$$

together with the Poincaré algebra itself (equations (0.1) and (0.3)).

(5.a) Define the supercommutator:

$$[\mathcal{O}_a, \mathcal{O}_b\} \equiv \mathcal{O}_a\mathcal{O}_b - (-1)^{\epsilon_a\epsilon_b}\mathcal{O}_b\mathcal{O}_a \ , \tag{0.18}$$

where $\epsilon_a \in \{0,1\}$ is the $\mathbb{Z}_2$ grading of $\mathcal{O}_a$. Using the Jacobi identities of a super-algebra:

$$(-1)^{\epsilon_c\epsilon_a}[[\mathcal{O}_a, \mathcal{O}_b\}, \mathcal{O}_c\} + (-1)^{\epsilon_a\epsilon_b}[[\mathcal{O}_b, \mathcal{O}_c\}, \mathcal{O}_a\} + (-1)^{\epsilon_b\epsilon_c}[[\mathcal{O}_c, \mathcal{O}_a\}, \mathcal{O}_b\} = 0 \ ,$$

show that the 4d $\mathcal{N}=1$ supersymmetry algebra closes.

Hint: For this computation, a useful identity is:

$$\sigma^\mu\bar\sigma^\nu\sigma^\rho + \sigma^\rho\bar\sigma^\nu\sigma^\mu = 2(\eta^{\mu\rho}\sigma^\nu - \eta^{\nu\rho}\sigma^\mu - \eta^{\mu\nu}\sigma^\rho) \ . \tag{0.19}$$

(5.b) Write down the supersymmetry algebra in terms of the Majorana spinor:

$$(\mathbf{Q}_a) = \begin{pmatrix} Q_\alpha \\ \bar Q^{\dot\alpha} \end{pmatrix} \ . \tag{0.20}$$

6. The 2d Poincaré supersymmetry algebra.

(This last problem is optional, but it should be fun to think about.)

In two dimensions, we can have left- and right-moving real supercharges, Q_+^I and Q_-^J, with $I = 1, \cdots, n_+$, and $J = 1, \cdots, n_-$, which generate the so-called $\mathcal{N} = (n_+, n_-)$ supersymmetry algebra.

(6.1) Write 2d vectors in terms of bi-spinors in Lorentzian signature, similarly to what we did in 4d.

(6.2) Write down the algebra of 2d $\mathcal{N} = (1, 1)$ supersymmetry (and justify the answer). What about the other cases, for instance $\mathcal{N} = (2, 0)$?