# Supersymmetry & Supergravity: Problem sheet 3

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*Due by Thursday, week 5 (February 14th), 5pm.*

## 1. Supersymmetry variations of a chiral multiplet.

Recall the supersymmetry variations for the chiral and anti-chiral multiplets:

$$\begin{aligned}
&\delta\phi = \sqrt{2}\epsilon\psi \ , && \delta\bar\phi = \sqrt{2}\bar\epsilon\bar\psi \ , \\
&\delta\psi_\alpha = i\sqrt{2}(\sigma^\mu\bar\epsilon)_\alpha\partial_\mu\phi + \sqrt{2}\epsilon_\alpha F \ , && \delta\bar\psi^{\dot\alpha} = i\sqrt{2}(\bar\sigma^\mu\epsilon)^{\dot\alpha}\partial_\mu\bar\phi + \sqrt{2}\bar\epsilon^{\dot\alpha}\bar F \ , \\
&\delta F = i\sqrt{2}\bar\epsilon\bar\sigma^\mu\partial_\mu\psi \ , && \delta\bar F = i\sqrt{2}\epsilon\sigma^\mu\partial_\mu\bar\psi \ .
\end{aligned} \tag{0.1}$$

Let us also write down the Lagrangian:

$$\mathscr{L}_{\rm kin} = -\partial_\mu\bar\phi\partial^\mu\phi - i\bar\psi\bar\sigma^\mu\partial_\mu\psi + \bar F F \ , \tag{0.2}$$

and:

$$\mathscr{L}_W = F^i\partial_i W - \frac{1}{2}\psi^i\psi^j\,\partial_i\partial_j W \ , \tag{0.3}$$

for $W = W(\phi)$ an arbitrary superpotential.

(1.a) By explicit computation using the SUSY variations (0.1), show that the Lagrangian $\mathscr{L}_{\rm kin}$ is supersymmetric:

$$\delta\mathscr{L}_{\rm kin} = \partial_\mu(\cdots) \ . \tag{0.4}$$

(1.b) Similarly, show by explicit computation that the interaction Lagrangian $\mathscr{L}_W$ is supersymmetric.

## 2. Coset manifold: a classic example.

Consider the group $G = SU(2)$, with group elements given by:

$$g = e^{i\epsilon_i T^i} \ \in G \ , \qquad [T^i, T^j] = i\epsilon^{ij}{}_k T^k \ . \tag{0.5}$$

(Of course $i = 1, 2, 3$.) We consider the subgroup $H = U(1)$ with a single generator:

$$h = e^{i\varepsilon T^3} \ \in H \ . \tag{0.6}$$

Show explicitly that the coset manifold $G/H$ is the two-sphere:

$$\mathcal{M} = G/H \cong S^2 \ . \tag{0.7}$$

*Hint:* One can consider an explicit realisation of $SU(2)$ in terms of $T^i = \frac{1}{2}\sigma^i$, with $\sigma^i$ the Pauli matrices. Then, the general element $g$ takes the form:

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \ , \qquad a, b \in \mathbb{C} \ , \quad \text{such that } |a|^2 + |b|^2 = 1 \ . \tag{0.8}$$

(Check this.) Show that, in this parameterisation, the action $g \to gh$ is given by:

$$a \to a e^{i\frac{\varepsilon}{2}} \ , \qquad b \to b e^{-i\frac{\varepsilon}{2}} \ . \tag{0.9}$$

Use this to argue that the coset manifold is indeed $S^2$. Can you find a convenient set of coordinates on the coset? How does $G = SU(2)$ act on those coordinates?

## 3. Rotations in superspace.

Consider 4d $\mathcal{N} = 1$ superspace:

$$\mathbb{R}^{3,1|4} = ISO(1,3|4)/SO(1,3) \ , \tag{0.10}$$

in the notation of the lecture notes. Recall that supersymmetry is given by an explicit translation in superspace:

$$e^{i(\eta Q + \bar{\eta}\bar{Q})} \ : \ (x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) \to (x^\mu - i\eta\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\bar{\eta}, \theta^\alpha + \eta^{\dot{\alpha}}, \bar{\theta}^{\dot{\alpha}} + \bar{\eta}^{\dot{\alpha}}) \ . \tag{0.11}$$

while ordinary translations are simply:

$$e^{ia^\mu P_\mu} \ : \ (x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) \to (x^\mu + a^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) \ . \tag{0.12}$$

From these transformations, we can write down the generators of supersymmetry and translation as differential operators on *superfields.*

(3.a) Recall that, on scalar fields in *Minkowski space-time*, the $SO(1,3)$ generators are given by $\mathbf{M}_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\nu)$. Give a physical explanation why this expression cannot be the correct realisation of $M_{\mu\nu}$ on scalar *superfields.*

(3.b) Compute the explicit action of:

$$g_R = e^{\frac{i}{2}\omega^{\mu\nu}M_{\mu\nu}}$$

on the superspace coordinates $y = (x, \theta, \bar{\theta})$, using the definition:

$$g_R^{-1}\mathbf{x}(y) = \mathbf{x}(y')h(g_R, y) \ .$$

In other words: find the analogue of (0.11) and (0.12) for $SO(1,3)$ rotations.

(3.c) Write down the superspace differential operator $\mathbf{M}_{\mu\nu}$ acting on scalar superfields. It should be of the form:

$$\mathbf{M}_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\nu) + \cdots$$

where the extra terms in the ellipsis are to be determined.

### 4. Manipulating Grassmann coordinates.

Let $\eta^i$ denote a set of $n$ Grassmann coordinates ($i = 1, \cdots, n$), which satisfy the Grassmann algebra:

$$\{\eta^i, \eta^j\} = 0 \ . \tag{0.13}$$

Let us define the integration over the $\eta$ coordinates as:

$$\int d^n\eta = \int d\eta^n \cdots d\eta^2 d\eta^1 \ . \tag{0.14}$$

(4.a) Prove that:

$$\int d^n\eta \, e^{\frac{1}{2}A_{ij}\eta^i\eta^j} = \mathrm{Pf}(A) \ ,$$

where $A_{ij} = A_{ji}$ is an antisymmetric matrix.

(4.b) For a single variable $\eta$, let us define the Dirac delta function $\delta(\eta - \theta)$ by:

$$\int d\eta \, \delta(\eta - \theta) f(\eta) = f(\theta) \ , \qquad \forall f \ . \tag{0.15}$$

Show that $\delta(\eta - \theta) = \eta - \theta$.

### 5. Chiral superfields.

Consider the explicit form:

$$\begin{aligned} \Phi(x, \theta, \bar\theta) &= \phi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) \\ &\quad + i\theta\sigma^\mu\bar\theta\partial_\mu\phi(x) + \frac{i}{\sqrt{2}}\theta\theta\bar\theta\bar\sigma^\mu\partial_\mu\psi(x) + \frac{1}{4}\theta\theta\bar\theta\bar\theta\partial^2\phi(x) \ . \end{aligned} \tag{0.16}$$

for the chiral superfield, and the differential operators:

$$\begin{aligned} &\mathbf{Q}_\alpha = -i\left(\partial_\alpha - i(\sigma^\mu\bar\theta)_\alpha\partial_\mu\right) \ , && \bar{\mathbf{Q}}_{\dot\alpha} = i\left(\bar\partial_{\dot\alpha} - i(\theta^\alpha\sigma^\mu)_{\dot\alpha}\partial_\mu\right) \ , \\ &\mathrm{D}_\alpha = \partial_\alpha + i(\sigma^\mu\bar\theta)\partial_\mu \ , && \bar{\mathrm{D}}_{\dot\alpha} = \bar\partial_{\dot\alpha} + i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu \ . \end{aligned} \tag{0.17}$$

(5.a) By explicit computation, check that:

$$\bar{\mathrm{D}}_{\dot\alpha}\Phi = 0 \ .$$

(5.b) Rederive the chiral multiplet supersymmetry transformation laws (0.1) using the superspace definition:

$$\delta\Phi = i(\epsilon\mathbf{Q} + \bar\epsilon\mathbf{Q})\Phi \ .$$

(5.c) Consider a (spinor-valued) superfield $\mathcal{N}_\alpha$ defined out of a general superfield $\mathcal{S}$ by:

$$\mathcal{N}_\alpha \equiv \bar{\mathrm{D}}\bar{\mathrm{D}}D_\alpha\mathcal{S} = \bar{\mathrm{D}}_{\dot\beta}\bar{\mathrm{D}}^{\dot\beta}D_\alpha\mathcal{S} \ . \tag{0.18}$$

Prove that $\mathcal{N}_\alpha$ is a chiral superfield.

## 6. Supersymmetric gauge transformations.

Given a real superfield $\mathcal{S}$ ($(\mathcal{S})^\dagger = \mathcal{S}$), we may define the transformation:

$$\mathcal{S} \to \mathcal{S} + \Phi + \bar{\Phi} \ , \tag{0.19}$$

where $\Phi$ and $\bar{\Phi}$ are a chiral multiplet and its Hermitian conjugate.

(6.a) Write down this transformation in terms of the components fields of the real superfields,

$$\mathcal{S} = \big(C, \chi \ , \bar{\chi} \ , M \ , \bar{M} \ , v_\mu \ , \lambda \ , \bar{\lambda} \ , D\big) \ ,$$

and of the component fields of $\Phi$ and $\bar{\Phi}$. (For instance, for the bottom component, we obviously have: $C \to C + \phi + \bar{\phi}$.) Why is this called a "gauge transformation"?

(6.b) Show that (0.19) leaves the D-term action:

$$S = \int d^4x \int d^2\theta d^2\bar{\theta}\, \mathcal{S} \tag{0.20}$$

invariant.

## 7. Superspace $\mathbb{R}^{1|1}$.

Consider minimally supersymmetric quantum mechanics (1d $\mathcal{N} = 1$). The supersymmetry algebra is simply:

$$\{Q, Q\} = 2H \ , \tag{0.21}$$

with $H$ the Hamiltonian. This is a simple toy model for superspace techniques, albeit slightly degenerate. There is no Lorentz group, so superspace $\mathbb{R}^{1|1}$ is simply the set of coordinates $(t, \theta)$, with the time coordinate $t$ and a single Grassmannian direction $\theta$. The superfields have the form:

$$\mathcal{S} = f_0(t) + \theta f_1(t) \ . \tag{0.22}$$

(7.a) Develop the theory of 1d $\mathcal{N} = 1$ superspace following our discussion 4d $\mathcal{N} = 1$ supersymmetry. What is the action of supersymmetry on 1d superspace? Check that the corresponding differential operators $\mathbf{Q}$ and $\mathbf{H}$ satisfy the supersymmetry algebra (0.21) on 1d superfields.

(7.b) Develop the theory of 1d $\mathcal{N} = 1$ superfields for 1d bosons and fermions. In particular, write down the real multiplet $(X, \psi)$ of the lecture notes (section 1.3) as a superfield. You should introduce the "scalar superfields" and "fermi superfields:"

$$\Phi^i = \phi^i + i\theta\psi^i \ , \qquad \Lambda^a = \lambda^a + 2\theta G^a \ , \tag{0.23}$$

where $\phi^i$ are dynamical boson, $\psi^i$ and $\lambda^a$ are dynamical fermions, and $G^a$ are auxiliary fields.

(7.c) Show how to write down supersymmetric actions systematically. Can you construct an example of an interacting 1d model with $\mathcal{N} = 1$ supersymmetry?