Supersymmetry & Supergravity: Problem sheet 4

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Due by Thursday, week 7 (February 28th), 5pm.

1. Supersymmetric vacua: WZ models.

Analyse the vacuum structure of the following 4d $\mathcal{N}=1$ supersymmetric theories of chiral multiplets. You can assume a canonical Kähler potential. What is the energy of the vacuum, in each case?

(1.a) A theory of two chiral multiplets X and Y, with superpotential:

$$W = \lambda X^2 Y + \mu X^2 \ , \tag{0.1}$$

with λ, μ some non-zero coupling constants.

(1.b) A theory of three chiral multiplet X, Y and Z, with superpotential:

$$W = \alpha Y + \beta Y X^2 + \gamma X Z \ , \tag{0.2}$$

with $\alpha, \beta, \gamma \neq 0$. What happens when $\alpha = 0$?

(1.c) A theory of a single chiral multiplet X, with superpotential:

$$W = \alpha X + \frac{\beta}{X} \ , \tag{0.3}$$

with $\alpha, \beta \neq 0$.

2. Vector multiplets.

In the WZ gauge, the $U(1)$ vector multiplet reads:

$$V = -\theta\sigma^\mu\bar{\theta} A_\mu + i\theta\theta\bar{\theta}\bar{\lambda} - i\bar{\theta}\bar{\theta}\theta\lambda + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}\, D \ . \tag{0.4}$$

The field-strength chiral multiplet is defined as:

$$\mathcal{W}_\alpha = -\frac{i}{4}\bar{\mathrm{D}}\bar{\mathrm{D}}\mathrm{D}_\alpha V \ . \tag{0.5}$$

(2.a) By explicit computation, show that the superfield $\mathcal{W}$ defined as in eq.(0.5) is given by:

$$\mathcal{W}_\beta = \lambda_\beta(z) - \theta^\alpha\Big((\sigma^{\mu\nu})_{\alpha\beta}F_{\mu\nu}(z) + i\epsilon_{\alpha\beta}D(z)\Big) + i\theta\theta(\sigma^\mu\partial_\mu\bar{\lambda}(z))_\beta\ , \qquad (0.6)$$

in terms of the chiral coordinate $z^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$.

(2.c) Compute the F-term Lagrangian:

$$\mathscr{L}_{\mathcal{W}^2} = -\frac{1}{2}\int d^2\theta \mathcal{W}^\alpha\mathcal{W}_\alpha\ , \qquad (0.7)$$

in field components.

(2.b) Compute the D-term Lagrangian:

$$\mathscr{L}_{\bar{\Phi}\Phi} = \int d^2\theta d^2\bar{\theta}\,\bar{\Phi}e^{-2V}\Phi\ , \qquad (0.8)$$

with Φ a chiral multiplet and V the vector multiplet in WZ gauge, in field components. You should find:

$$\mathscr{L}_{\bar{\Phi}\Phi} = \mathscr{L}_0 + A_\mu j^\mu + A_\mu A^\mu X + Y(D,\lambda,\bar{\lambda})\ , \qquad (0.9)$$

for $\mathscr{L}_0$ the kinetic term without gauge field, some j^μ and X given in terms of the chiral superfield components, and a term Y that depends on D and the gaugino. Write the answer in terms of the gauge-covariant derivative.

(2.d) Generalise all the above superfield computations to the non-abelian case.

3. SQCD with $N_f = N_c + 1$.

Consider SQCD with gauge group $SU(N_c)$ and $N_f = N_c + 1$ flavors, $\widetilde{Q}^i, Q_j$, the "squark" fields, and zero superpotential.

(3.a) Write down all the gauge-invariant chiral superfields that one can construct from the flavors (matter fields in chiral superfields, in the fundamental and anti-fundamental of $SU(N_c)$), the "mesons and baryons," denoted by:

$$X = (M,\, B,\, \widetilde{B})\ , \qquad (0.10)$$

schematically. Check that there are $N_c^2 + 4N_c + 3$ such fields X.

(3.b) In which representations of the flavor group:

$$G_F = SU(N_f)_+ \times SU(N_f)_- \times U(1)_B\ ,$$

with $N_f = N_c + 1$, do the fields (0.10) transform? What is the symmetry group of this theory in the special case $N_c = 2$?

(3.c) As explained in the lectures, the non-anomalous R-symmetry of SQCD (with $N_f = N_c + 1$, here) is given by:

$$R[\widetilde{Q}^i] = R[Q_j] = r = 1 - \frac{N_c}{N_f} = \frac{1}{N_c + 1} \ . \tag{0.11}$$

Write down a table giving the representations (or charges) under $G_F \times U(1)_R$, in $N_f = N_c + 1$ SQCD, for all the *fermions* in the theory.

(3.d) Write down a similar table for the fermions in a theory of chiral multiplets consisting only of the gauge-invariant chiral superfields X that you built in (0.10). We call this theory (with only the "mesons and baryons" as fundamental fields, and no gauge fields) the "candidate IR theory."

(3.e) Using the result of question (3.c), compute the following 't Hooft anomalies for the UV theory, $N_f = N_c + 1$ SQCD:

$$\begin{aligned} &\mathrm{Tr}(SU(N_f)^3_\pm) \ , \\ &\mathrm{Tr}(SU(N_f)^2_\pm\, U(1)_B) \ , \\ &\mathrm{Tr}(SU(N_f)^2_\pm\, U(1)_R) \ , \\ &\mathrm{Tr}(U(1)_B^2 U(1)_R) \ , \\ &\mathrm{Tr}(U(1)_R^3) \ , \\ &\mathrm{Tr}(U(1)_R) \ . \end{aligned} \tag{0.12}$$

E.g.: for the first one, we get: $\mathrm{Tr}(SU(N_f)^3_+) = -N_c$ from the chiral multiplets Q_j, which are in the anti-fundamental of $SU(N_f)_+$.

(3.f) Using the result of question (3.d), compute the same 't Hooft anomalies for the candidate IR theory, and compare it to the ones obtained in (3.e). What does your result suggest?

(3.g) Argue that, for the candidate IR theory, the only superpotential compatible with the $G_F \times U(1)_R$ global symmetry is of the form:

$$W = \alpha f(X) \ , \tag{0.13}$$

with some function $f(X)$ and some overall dimensionfull coupling α. Find $f(X)$, and give α as a power of the dynamical QCD scale, $\alpha \sim \Lambda^n$ for some $n \in \mathbb{Z}$.

(3.h) [This question is optional.] Write down the vacuum equations for this W and analyse them. How does the vacuum moduli space compare to the one of *classical* SQCD with $N_f = N_c + 1$? Assuming that the two should coincide, can you fix any remaining undetermined constant in $f(X)$?

4. Supersymmetry breaking and goldstino.

In this problem, we explore F-term spontaneous supersymmetry breaking. Consider a general theory of n chiral multiplets Φ^i, with canonical Kähler potential and arbitrary superpotential $W(\Phi)$. Assume that supersymmetry is spontaneously broken, so that there exists a (classical) vacuum with a non-zero n-vector:

$$\bar{f}_i \equiv \frac{\partial W}{\partial \phi^i} \neq 0 \ , \tag{0.14}$$

where f^i is the VEV of the auxiliary field F^i on-shell.

(4.a) The classical vacuum is defined as:

$$\frac{\partial V_0}{\partial \phi^i} = 0 \ , \qquad \frac{\partial V_0}{\partial \bar{\phi}_i} = 0 \ . \tag{0.15}$$

Using the classical Lagrangian for this theory, show that eq.(0.14) implies the existence of a *massless fermion* in the vacuum. This is the *goldstino*, the analogue of a goldstone boson for spontaneously broken supersymmetry.

(4.b) Compute the masses of the fermionic and bosonic excitations just above the vacuum, and show that:

$$\mathrm{STr}(M^2) = 0 \ . \tag{0.16}$$

In other words, in terms of *mass eigenstates*, the sum of the boson masses squared *minus* the sum of the fermion masses squared vanishes.

(4.c) Illustrate these general results explicitly in the model of problem (1.b) with superpotential (0.2).