Practical Numerical Analysis: Sheet 4

1. Compute numerical solutions to the Curtiss-Hirschfelder differential equation

$$u'(t) \;=\; f(t,u) \;=\; -50(u - \cos t)\ , \quad 0 \le t \le 10\ ,$$

with $u(0) = 1$, using a uniform timestep of $\Delta t = 0.01$ and the following methods:

(a) explicit Euler

(b) implicit Euler

(c) Crank Nicolson

(d) improved Euler

(e) RK4

The exact solution is

$$u(t) \;=\; \frac{2500}{2501}\cos t + \frac{50}{2501}\sin t + \frac{1}{2501}\mathrm{e}^{-50t}\ .$$

Compare the errors for each method.

For the Crank Nicolson scheme, compute how many steps are required theoretically (with justification) in order to compute the solution to within a tolerance of 10^{-6} of the exact solution at each timestep. In practice, what is the error with this number of timesteps?

2. Use the implicit Euler method to solve the van der Pol equation

$$u'' - \epsilon(1-u^2)u' + u \;=\; 0;\ ,$$

with $u(0) = u'(0) = 1/2$ and $\epsilon = 1, 5$ on the interval $[0, 20]$ and using 10001 equally spaced time points. Plot the solutions as a function of time. (First write the equation as a first order system and use Newton's method to solve the nonlinear system.)

3. Use one of the adaptive Runge Kutta schemes discussed in lectures to compute an accurate solution to the differential equation

$$u'(t) \;=\; \frac{12\pi}{(11-2t)^2}\cos\left(\frac{6\pi}{11-2t}\right)\mathrm{e}^{-(11-2t)/2} + u\ ,$$

with $u(0) = \sin(6\pi/11)\mathrm{e}^{-11/2}$. The exact solution to the problem is

$$u(t) \;=\; \sin\left(\frac{6\pi}{11-2t}\right)\mathrm{e}^{-(11-2t)/2}\ .$$

Further Reading

1. E. Süli and D. Mayers, *An Introduction to Numerical Analysis*, CUP, 2003.

2. A. Iserles, *A First Course in the Numerical Analysis of Differential Equations*, Cambridge Texts in Applied Mathematics, CUP, 1996.