Practical Numerical Analysis: Sheet 5

1. Sometimes when we wish to test numerical methods for PDEs we use the so-called *Method of Manufactured Solutions* (MMS) in which we think up a solution $u(x, y)$ to a problem and the work out the associated problem. In this question we wish to solve Poisson's equation in the unit square, $\Omega = [0, 1]^2$ with homogenous Dirichlet boundary conditions. To do this we choose a problem with solution

$$u(x, y) \quad = \quad x^2(1 - x)\sin(\pi y) \ .$$

Note that this has homogeneous Dirichlet boundary conditions on $\partial\Omega$. By differentiating $u(x, y)$ we find that this is the solution to Poisson's equation in the form

$$-\nabla^2 u \quad = \quad f(x, y) \quad = \quad (\pi^2 x^2(1 - x) + 6x - 2)\sin(\pi y) \ .$$

As in lectures, use a uniform mesh spacing of size h in both the x and y directions and then use a finite difference scheme to write the problem as a matrix equation $A\mathbf{U} = \mathbf{f}$. Recall that A has a block structure. There are many ways to construct A but one way is described by the function `makeA.m` below.

```
function A=makeA(N)
e=ones(N-1,1);
B=spdiags([-e 4*e -e], -1:1, N-1, N-1);
C=-speye(N-1);
A=sparse((N-1)^2,(N-1)^2);
for i=1:N-1
    A((i-1)*(N-1)+1:i*(N-1),(i-1)*(N-1)+1:i*(N-1))=B;
end
for i=1:N-2
    A((i-1)*(N-1)+1:i*(N-1),i*(N-1)+1:(i+1)*(N-1))=C;
    A(i*(N-1)+1:(i+1)*(N-1),(i-1)*(N-1)+1:i*(N-1))=C;
end
```

Either using this function, or one you construct yourself, compute the finite difference approximation to Poisson's equation above using N = 8, 16, 32 and 64 mesh spacings in each direction, solving the linear system using backslash, and confirm that the maximum pointwise error converges as h^2. (You may find the `reshape` command helpful.)

P.T.O.

2. Solve the same problem as in question 1 but this time, instead of using backslash to solve the linear system, use the following splitting methods:

   (a) Jacobi

   (b) Gauss-Seidel

   (c) SOR

   In this example you should fix $N = 32$ and use a stopping tolerance of 10^{-8}. How many iterations do each of the methods take to converge? For Jacobi's method demonstrate that the predicted rate of convergence of the iterates is achieved in practice. You may choose any initial guess for the iterative methods but you should be clear what this guess is and use the same guess for each of the three methods.