# Epidemic Models

Compartment models:

$$S \longrightarrow I \longrightarrow R$$

susceptible infective removed (cured, acquire immunity, or die)

Assume Law of Mass Action — rate of reaction directly proportional to product of (active) reactant concentrations (or densities).

Since $S$ & $I$ interact, really we have

$$S + I \xrightarrow{\beta} I$$

$$I \xrightarrow{\gamma} R$$

$$\therefore \quad \frac{dS}{dt} = -\beta S I \qquad (1)$$

$$\frac{dI}{dt} = \beta S I - \gamma I \qquad (2)$$

$$\frac{dR}{dt} = \gamma I \qquad (3)$$

Initial cond$^{ns}$ $S(0)=S_0$, $I(0)=I_0$, $R(0)=0$,
$S_0$, $I_0$ both greater than 0.

This is the Kermack-McKendrick 1927 model.

<u>NB 1.</u> Add all 3 eqns:

$$\frac{d}{dt}(S+I+R)=0$$

$$\therefore \quad S+I+R=N \text{ (const)} = I_0+S_0.$$

<u>NB 2</u> : $$\frac{dI}{dt}=\beta[S-\rho]I \quad \text{where } \rho\equiv\frac{\gamma}{\beta}$$

If $S(0)<\rho$, then $\frac{dI}{dt}<0 \ \forall t\geq 0$
(Since $\frac{dS}{dt}\leq 0 \ \forall t \ \therefore S(t)<\rho \ \forall t$).

<u>Defn</u>. Epidemic means that $I(t)>I_0$ for some $t>0$.

So, $S(0)<\rho \Rightarrow$ no epidemic
$S(0)>\rho \Rightarrow$ epidemic

Defn. $\rho = \frac{\gamma}{\beta}$ is called the relative removal rate, $\sigma = \frac{1}{\rho}$ is the infectious contact rate, $R_0 = \sigma S_0$ is the basic reproduction rate of infection — it is the number of secondary infections produced by one primary infection in a wholly susceptible pop$^n$.

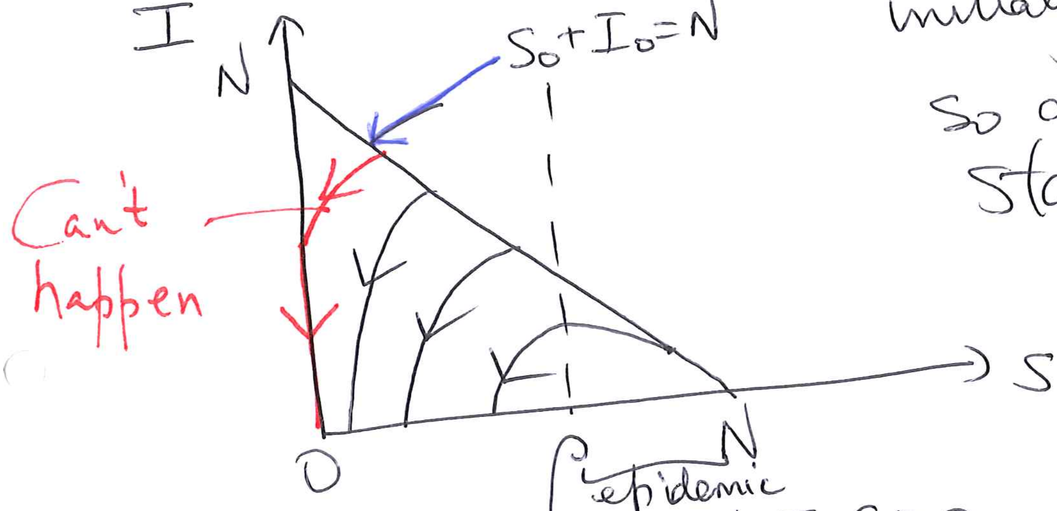
[ $\beta S_0$ is the # of secondary infections per unit time produced by one infective. Infectives move out of infection at rate $\gamma$, ∴ they are infective for time $\frac{1}{\gamma}$ ∴ they produce $\frac{1}{\gamma}\beta S_0$ secondary infections ]

NB3
$$\left.\frac{dI}{dt}\right|_{t=0} = \gamma(\sigma S_0 - 1)\, I(t)\Big|_{t=0} \quad \text{from (2)}$$
$$= \gamma(R_0 - 1)\, I(t)\Big|_{t=0}$$

$\Rightarrow$ $R_0 > 1 \Rightarrow$ epidemic
$R_0 < 1 \Rightarrow$ no epidemic

Eqn (3) decouples from (1) & (2).

Let us draw phaseplane for (1) & (2):




initially $S_0+I_0=N$, so all trajectories start on this line

$\forall\, I, S>0, \quad \dot{S}<0$

For $S>\rho \quad \dot{I}>0$
$\phantom{For\ } S<\rho \quad \dot{I}<0$

Also, $\quad I+S+R=N \quad \forall t$

$\Rightarrow \quad I+S \leq N \quad \forall t \quad$ since $R(t)\geq 0$

$\left[\left.\frac{dR}{dt}\right|_{t=0} = \gamma I\Big|_{t=0}\right.$

$\therefore$ R goes +ve.
For R to go −ve, $I<0$, but
$\frac{dI}{dt}=(\beta S-\gamma)I$
and for $I=0$, $\frac{dI}{dt}=0$ $\therefore$
if $I(0)>0$, $\quad I(t)\geq 0 \quad \forall t\geq 0\Big]$

$\therefore$ curves must be in upper quadrant but below
$I+S=N$

NB4 Red curve cannot exist!!

From (1) & (3): $\dfrac{dS}{dR} = -\dfrac{S}{\rho}$

$\Rightarrow \quad S = S_0 e^{-R/\rho} \geq S_0 e^{-\frac{N}{\rho}}$ since $R\leq N$
$> 0 \quad \forall t$

∴ the infection dies out due to lack of infectives (not due to lack of susceptibles).

## Venereal Diseases.
(Hethcote and Yorke 1984: Gonorrhea Transmission Dynamics and Control. Lect. Notes in Biomath. 56, Springer.)

Transmission of STD e.g. gonorrhea, chlamydia, syphilis, AIDS is a major problem. [eg in USA more than 2 million ($\approx 1$ percent) contact gonorrhea / year, more than 4 million acquire chlamydia (can lead to sterility in women)

N.B. No acquired immunity (i.e. no R class).
N.B. Restricted to sexually active part of the community.
N.B. No overt symptoms to quite late in infection.

## A Model for Gonorrhea Transmission.

Assumptions: – The pop$^n$ is uniformly promiscuous

- Heterosexual encounters
- "Criss-cross" disease (only males can infect females and vice-versa)
- homogeneous mixing (no pair formation) [really 1st assumption]
- incubation period comparatively short (3-7 days) (compared to length of infection).

Schematically:

$$\begin{tikzcd} S \arrow[r] & I \arrow[l, bend right] \arrow[dl] \\ \bar{S} \arrow[r] & \bar{I} \arrow[l, bend left] \arrow[ul] \end{tikzcd}$$

– female

$$S(t) + I(t) = N, \quad \bar{S}(t) + \bar{I}(t) = \bar{N}$$

The equations are:

$$\frac{dS}{dt} = \underset{\text{infection}}{-rS\bar{I}} + \underset{\text{recovery}}{aI} \qquad \frac{d\bar{S}}{dt} = -\bar{r}\bar{S}I + \bar{a}\bar{I}$$

$$\frac{dI}{dt} = rS\bar{I} - aI \qquad \frac{d\bar{I}}{dt} = \bar{r}\bar{S}I - \bar{a}\bar{I}$$

where $r, \bar{r}, a, \bar{a}$ are positive constants; $S(0) = S_0$, $I(0) = I_0$, $\bar{S}(0) = \bar{S}_0$, $\bar{I}(0) = \bar{I}_0$

Note that adding we have $S+I = N$, $\bar{S}+\bar{I} = \bar{N}$.

Note also that we may reduce the system by substituting $S = N - I$, $\bar{S} = \bar{N} - \bar{I}$ to get:

$$\frac{dI}{dt} = r(N-I)\bar{I} - aI$$

$$\frac{d\bar{I}}{dt} = \bar{r}(\bar{N}-\bar{I})I - \bar{a}\bar{I}$$

[Now we could analyse this system using phase-plane techniques]. The steady states (or equilibrium states) are:

$\frac{dI}{dt} = \frac{d\bar{I}}{dt} = 0 \Rightarrow I = \bar{I} = 0$ and $I = I^* = \frac{N\bar{N} - \rho\bar{\rho}}{\bar{N} + \rho}$,
$= \bar{I}^* = \frac{N\bar{N} - \rho\bar{\rho}}{\bar{\rho} + N}$ which exist iff. $\frac{N\bar{N}}{\rho\bar{\rho}} > 1$. – ie. threshold.
where $\rho = \frac{a}{r}$, $\bar{\rho} = \frac{\bar{a}}{\bar{r}}$.

Note that due to changing sexual habits $N, \bar{N}$ have increased

$\therefore$ $N\bar{N} > \rho\bar{\rho}$, therefore $(I^*, \bar{I}^*)$ exists.

Stability :

$(0,0)$ Linearise :
$$\frac{dI}{dt} = rN\bar{I} - aI$$
$$\frac{d\bar{I}}{dt} = \bar{r}\bar{N}I - \bar{a}\bar{I}$$

$\Rightarrow$ eigenvalue problem:
$$\begin{vmatrix} -a-\lambda & rN \\ \bar{r}\bar{N} & -\bar{a}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + (a+\bar{a})\lambda + a\bar{a} - r\bar{r}N\bar{N} = 0$$

$$\Rightarrow 2\lambda = -(a+\bar{a}) \pm \left[(a+\bar{a})^2 - 4(a\bar{a} - r\bar{r}N\bar{N})\right]^{1/2}$$

$$\Rightarrow 2\lambda = -(a+\bar{a}) \pm \left[(a+\bar{a})^2 + 4a\bar{a}\left(\frac{N\bar{N}}{\rho\bar{\rho}} - 1\right)\right]^{1/2}$$

$\therefore$ if $N\bar{N} > \rho\bar{\rho}$, $(0,0)$ is unstable.

$(I^*, \bar{I}^*)$ :
$$\begin{vmatrix} -a-r\bar{I}^*-\lambda & rN \\ \bar{r}\bar{N} & -\bar{a}-\bar{r}I^*-\lambda \end{vmatrix} = 0$$

$\Rightarrow \mathrm{Re}\,\lambda < 0$ $\therefore$ $(I^*, \bar{I}^*)$ (when it exists) is stable.

N.B. $\frac{N\bar{N}}{\rho\bar{\rho}} > 1 \Rightarrow \left(\frac{rN}{\bar{a}}\right)\left(\frac{\bar{r}\bar{N}}{a}\right) > 1$

Maximal male contact rate: If every male is susceptible

then $\frac{rN}{\bar{a}}$ is the av. no. of males contacted by a female infective during her infectious period : $\underbrace{\frac{1}{\bar{a}}}_{\text{av. infection period}} \times \underbrace{rN}_{\text{no. of males contacted if all}}$ [Typo Murray P. 622]

In the USA, 1973, $\frac{\bar{N}N}{\bar{\rho}\rho} \approx 1.127$, $\therefore (I^*, \bar{I}^*)$ exists.

$\therefore$ Suppose $N = 20m$, $\bar{N} = 20m$, then this $\Rightarrow$ $I^* = 1.12m$
$\bar{I}^* = 1.21m$.

## Multi-group Models.

$N_1$ - very active females — $N_2$ - very active males
$N_3$ - active females — $N_4$ - active Males
$\}$ asymptomatic when infectious

$N_5 - N_8$ corres. groups symptomatic when infectious.

Normalise : $N_1 + N_3 + N_5 + N_7 = 1$
$N_2 + N_4 + N_6 + N_8 = 1$
$S_i = 1 - I_i$

$$\underbrace{\frac{d}{dt}(N_i I_i)}_{\text{rate of new infectives}} = \underbrace{\sum_{j=1}^{8} L_{ij}(1 - I_i) N_j I_j}_{\text{rate of new infectives (incidence)}} - \underbrace{\frac{N_i I_i}{D_i}}_{\text{recovery rate of infectives}}$$

i.c. $I_i(0) = I_{i0}$. $(L_{ij})$ is contact matrix $L_{ij} = 0$ if $i+j$ even (i.e. disease spread by heterosexual contact only).

## Network Model (SIS)

Suppose there is a network of pop$^{ns}$

$$\frac{dS_i}{dt} = \underbrace{-\beta S_i I_i}_{\text{self-infection}} - \sum_{j \neq i} \tilde{\beta} a_{ij} S_i I_j + \gamma I_i$$

$a_{ij}$ → entries of the adjacency matrix

$$\frac{dI_i}{dt} = -\frac{dS_i}{dt}$$

$$a_{ij} = \begin{cases} 1 & \text{if } i \,\&\, j \text{ connected} \\ 0 & \text{otherwise} \end{cases}$$

N.B In gen, $\gamma$, $\beta$ will depend on $i$
& different rates of infection between pop$^{ns}$ can be taken into account by using a weighted adjacency matrix with entries $w_{ij}$ corresponding to rates of infection of pop$^n$ $i$ by pop$^n$ $j$

ex. 2 pop$^{ns}$.

$$\frac{dI_1}{dt} = \beta(1-I_1)I_1 + \tilde{\beta} a_{12}(1-I_1)I_2 - \alpha I_1 \qquad S_1 = 1-I_1$$

$$\frac{dI_2}{dt} = \beta(1-I_2)I_2 + \tilde{\beta} a_{21}(1-I_2)I_1 - \alpha I_2 \qquad S_2 = 1-I_2$$

st. st: $I_1 = 0 = I_2$ $\qquad J = \begin{pmatrix} \beta-\alpha & \tilde{\beta} \\ \tilde{\beta} & \beta-\alpha \end{pmatrix}$

$$\Rightarrow (\lambda-(\beta-\alpha))^2 - \tilde{\beta}^2 = 0$$

$$\Rightarrow \lambda = \beta - \alpha \pm \tilde{\beta}$$

$\Rightarrow$ unstable if $\tilde{\beta} > (\alpha - \beta)$
$\therefore$ if $\alpha > \beta$, then in isolation, the zero infection is stable, but with suff. strong connection it can be destabilised!

With more nodes (pop$^{ns}$), lin stab depends on details of the adj matrix (its structure).

## Ex. Model with migration

$$\frac{dS_i}{dt} = -\beta S_i I_i + \sum_{j \neq i} (m_{ij} S_j - m_{ji} S_i) + \gamma I_i$$

$$\frac{dI_i}{dt} = \beta S_i I_i + \sum_{j \neq i} (m_{ij} I_j - m_{ji} I_i) - \gamma I_i$$

where $m_{ij}$ is the migration rate from $i$ to $j$.