Swing

Aim & Questions

Simplifying Assumptions

(A1) The "swing-rider" system is approximated by a pendulum with variable length $l(t)$.

(A2) The rope is taken to be massless, & the rider is represented by a particle of mass m at the end of the rope (standing will move the mass up, hence $l(t)$ becomes smaller).

(A3) Air resistance is the only other force (as well as tension in the rope) – ie we ignore friction at the pivot.

(A4) System is modelled in 2D.

(A5) Rope is modelled as a rigid rod (light rod)

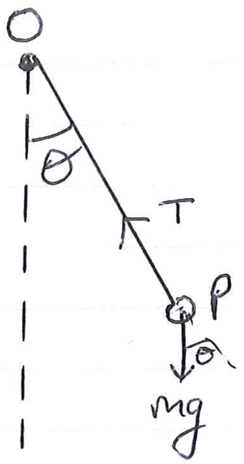


Figure S1.

Equations of Motion (Ignore Air Resistance) Assume $l(t)$ fixed

Method 1 : Newton's II Law.

Use (r, θ) coords. So P has coords $l(t)\,\underline{e}_r$
$= L\underline{e}_r$

[Recall $\underline{e}_r = \sin\theta\,\underline{i} - \cos\theta\,\underline{j}$
$\underline{e}_\theta = +\cos\theta\,\underline{i} + \sin\theta\,\underline{j}$
$\dot{\underline{e}}_r = \dot\theta\,\underline{e}_\theta, \quad \dot{\underline{e}}_\theta = -\dot\theta\,\underline{e}_r$]

"$\underline{F} = m\underline{a}$" $\Rightarrow (mg\cos\theta - T)\,\underline{e}_r - mg\sin\theta\,\underline{e}_\theta = m\ddot{\underline{r}}$ (1.1)

$\dot{\underline{r}} = L\dot\theta\,\underline{e}_\theta$
$\ddot{\underline{r}} = L\ddot\theta\,\underline{e}_\theta + L(-\dot\theta)^2\,\underline{e}_r$

$\therefore\; mg\cos\theta - T = -mL\dot\theta^2$ (1.2)
$-mg\sin\theta = mL\ddot\theta$ [or $\frac{d}{dt}(L^2\dot\theta) = -gL\sin\theta$] (1.3)

Suppose $L = l(t)$ (not fixed) : $\dot{\underline{r}} = \dot{l}\,\underline{e}_r + l\dot\theta\,\underline{e}_\theta$
$\ddot{\underline{r}} = \ddot{l}\,\underline{e}_r + 2\dot{l}\dot\theta\,\underline{e}_\theta + l\dot\theta^2(-\underline{e}_r) + l\ddot\theta\,\underline{e}_\theta$

$\therefore$ From (1.1) : $mg\cos\theta - T = m\ddot{l} - ml\dot\theta^2$ (1.4)
$-\cancel{m}g\sin\theta = +2\cancel{m}\dot{l}\dot\theta + \cancel{m}l\ddot\theta$ (1.5)

(1.5) $\Rightarrow \frac{d}{dt}(l^2\dot\theta) = -gl\sin\theta$

Method 2: Ang Mom.

$$(1.1) \Rightarrow m\ddot{\underline{r}} = -mg\sin\theta\, \underline{e}_\theta + (mg\cos\theta - T)\underline{e}_r$$

$$\cancel{m}\,\underline{r}\wedge\ddot{\underline{r}} = -\cancel{m}g\sin\theta\, \underline{e}_r \wedge l\underline{e}_\theta + 0 \qquad (1.6)$$

LHS of (1.6) is $\frac{d}{dt}(\underline{r}\wedge\dot{\underline{r}})$, so (1.6) becomes

$$\frac{d}{dt}(l^2\dot\theta)\, \underline{e}_r\wedge\underline{e}_\theta = -g\sin\theta\, l\, \underline{e}_r\wedge\underline{e}_\theta,$$

ie $$\frac{d}{dt}(l^2\dot\theta) = -gl\sin\theta \qquad (1.7)$$

(rate of change of ang mom) = torque (moment of external forces)

$$\left[\begin{array}{l} \underline{F} = m\ddot{\underline{r}} \\ \underline{r}\wedge\underline{F} = \underline{r}\wedge m\ddot{\underline{r}} = \frac{d}{dt}(\underline{r}\wedge m\dot{\underline{r}}) = \underline{r}\wedge\underline{F} \end{array}\right]$$

Method 3: Conservation of Energy

$\underline{F} = m\ddot{\underline{r}}$

$\Rightarrow \underline{F}.\dot{\underline{r}} = m\ddot{\underline{r}}.\dot{\underline{r}} = \frac{d}{dt}\left(\frac{m\dot{\underline{r}}^2}{2}\right)$

$\therefore$ Here, we have $\frac{md}{2dt}\left(\dot{l}^2 + l^2\dot{\theta}^2\right) = (mg\cos\theta - T)\dot{l} - mg\sin\theta\, l\dot{\theta}$ (1.8)

[N.B. If $\dot{l}=0$ then we have $\frac{d}{dt}\frac{ml^2\dot{\theta}^2}{2} = \frac{d}{dt}(mgl\cos\theta)$]

So, we have the unknown $mg\cos\theta - T$.

But (1.4) $\Rightarrow$ this can be replaced by $m\ddot{l} - ml\dot{\theta}^2$

$\therefore$ (1.8) becomes

$$\frac{1}{2}\frac{d}{dt}\left(\dot{l}^2 + l^2\dot{\theta}^2\right) = \dot{l}\ddot{l} - \dot{l}l\dot{\theta}^2 - g\sin\theta\, l\dot{\theta}$$

ie

$$\cancel{\dot{l}\ddot{l}} + \frac{2}{2}\dot{l}l\dot{\theta}^2 + \frac{2}{2}l^2\dot{\theta}\ddot{\theta} = \cancel{\dot{l}\ddot{l}} - \cancel{\dot{l}l\dot{\theta}^2} - g\sin\theta\, l\dot{\theta}$$

$$2l\dot{l}\dot{\theta}^2 + l^2\dot{\theta}\ddot{\theta} = -g\sin\theta\, l\dot{\theta}$$

$$2l\dot{l}\dot{\theta} + l^2\ddot{\theta} = -gl\sin\theta$$

ie $\frac{d}{dt}\left(l^2\dot{\theta}\right) = -gl\sin\theta$ (1.9)

So, we eventually have $\frac{d}{dt}(l^2\dot{\theta}) = -gl\sin\theta$

Non-dimensionalise

Let $\hat{l} = \frac{l}{L_0}$, $\tau = \frac{t}{T_0}$ L_0, T_0 fixed.

$\frac{d}{dt} = \frac{d}{d\tau}\frac{1}{T_0}$

$\therefore$ we have $\frac{L_0^2}{T_0^2}\frac{d}{d\tau}(\hat{l}^2\dot{\theta}) = -gL_0\hat{l}\sin\theta$ $\quad \dot{} \equiv \frac{d}{d\tau}$

$\Rightarrow \frac{d}{d\tau}(\hat{l}^2\dot{\theta}) = -\frac{gT_0^2}{L_0}\hat{l}\sin\theta$

I.C. $\cancel{l(t)}$ $l(0) = L$, say, so set $L_0 = L$ so that $\hat{l}(0) = 1$.

$\therefore$ Set $T_0^2 = \frac{L_0}{g}$ (check $[\text{time}]^2 = \frac{[\text{length}]}{\frac{[\text{length}]}{[\text{time}]^2}}$)

Drop ^: $\frac{d}{d\tau}(l^2\dot{\theta}) = -l\sin\theta$ (1.10)

Phase Plane Analysis

Let us assume that l is const $\therefore l = 1$.

So (1.10) becomes $\ddot{\theta} = -\sin\theta$ (1.11)

Look for steady states.

$\dot{\theta} = \ddot{\theta} = 0 \Rightarrow \sin\theta_0 = 0 \Rightarrow \theta_0 = n\pi, \quad n = 0, 1, 2, \ldots$ etc.

Recast (1.11) as a system of two 1st order ODEs:

$$\begin{aligned} \dot{\theta} &= \varphi \\ \dot{\varphi} &= -\sin\theta \end{aligned} \qquad (1.12)$$

st.st. (θ, φ) are $(n\pi, 0)$ $n = 0, 1, 2, \ldots$

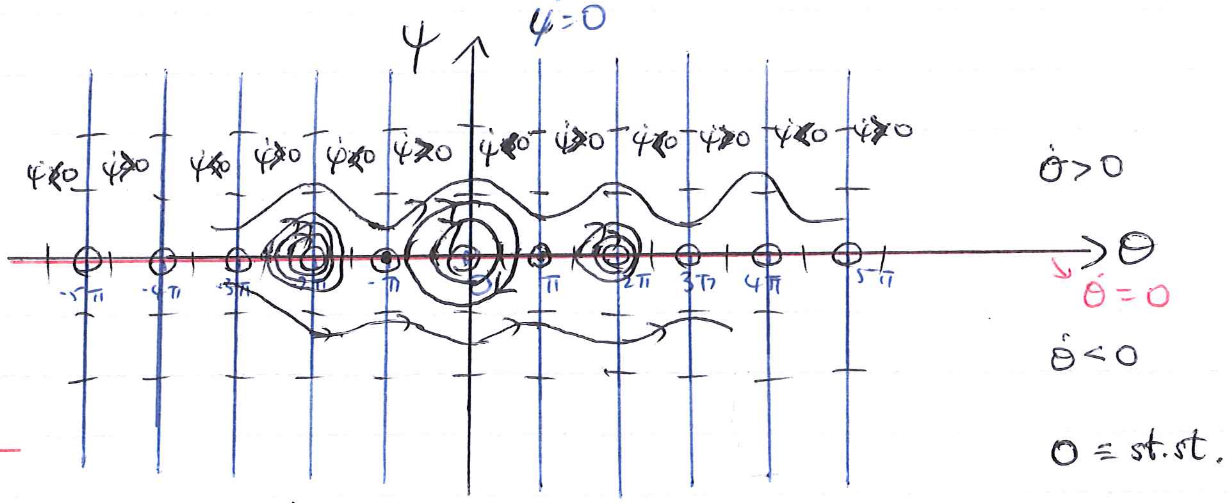


Figure S2

Nullclines: $\dot{\theta} = 0 \Rightarrow \varphi = 0$

Linear Stability Analysis: st.st $(2n\pi, 0)$: $\theta = \theta^* + \tilde{\theta}$, $\varphi = \tilde{\varphi}$

$\tilde{\theta}, \tilde{\varphi}$ small: $\dot{\tilde{\theta}} = \tilde{\varphi}$, $\dot{\tilde{\varphi}} = -\tilde{\theta}$

Jacobian is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \text{centre}$$

st. st. $((2n+1)\pi, 0)$: $\dot{\tilde{\theta}} = \tilde{\psi}$, $\dot{\tilde{\psi}} = \tilde{\theta}$

$$\Rightarrow \lambda^2 - 1 = 0 \quad \therefore \lambda = \pm 1 \Rightarrow \text{saddle}$$

What do these results mean?

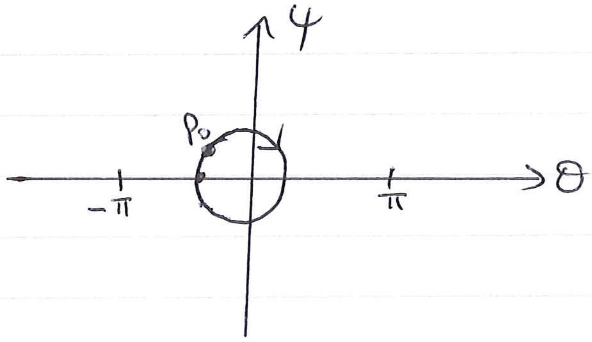
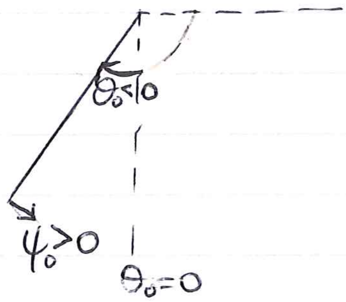


Figure S3

Int. Cond. $\theta = \theta_0$
$\psi = \psi_0$



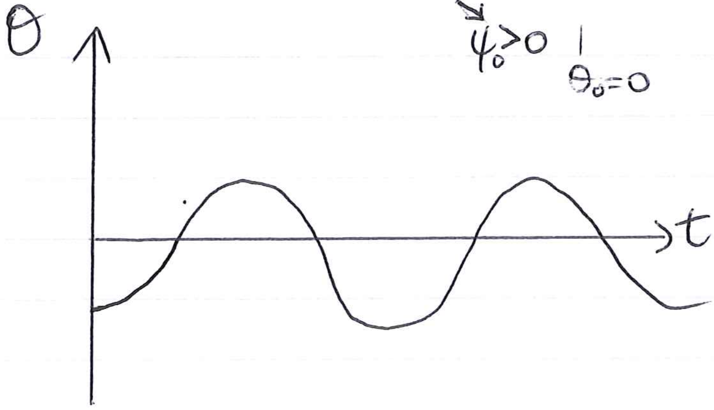


Figure S4

Is this realistic?