Linear Stability

$$\mathring{\theta} = f(\theta, \psi) \qquad (1)$$
$$\mathring{\psi} = g(\theta, \psi)$$

Defn. Steady state (θ^*, ψ^*) such that

$$f(\theta^*, \psi^*) = 0 = g(\theta^*, \psi^*).$$

Linear Stability:

Set $\theta = \theta^* + \hat{\theta}$, $|\hat{\theta}| \ll 1$
$\psi = \psi^* + \tilde{\psi}$ $|\tilde{\psi}| \ll 1$

Sub into (1): $\mathring{\hat{\theta}} = f(\theta^* + \hat{\theta}, \psi^* + \tilde{\psi})$

$$= \underbrace{f(\theta^*, \psi^*)}_{0 \text{ by defn of st. st.}} + f_\theta(\theta^*, \psi^*)\hat{\theta} + f_\psi(\theta^*, \psi^*)\tilde{\psi} + h.o.t.$$

$\therefore$ Obtain $$\begin{pmatrix} \mathring{\hat{\theta}} \\ \mathring{\tilde{\psi}} \end{pmatrix} = \begin{pmatrix} f_\theta & f_\psi \\ g_\theta & g_\psi \end{pmatrix} \begin{pmatrix} \hat{\theta} \\ \tilde{\psi} \end{pmatrix} \qquad (2)$$

where $J = \begin{pmatrix} f_\theta & f_\psi \\ g_\theta & g_\psi \end{pmatrix}$ is the Jacobian matrix & is evaluated at the st. st.

Soln to (2): Set $\begin{pmatrix} \hat{\theta} \\ \tilde{\psi} \end{pmatrix} = \underline{a} e^{\lambda \tau}$

$$\Rightarrow \lambda \underline{a} = J \underline{a}$$

$\Rightarrow (J - \lambda I)\underline{a} = 0$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore \lambda$ is an eigenvalue of J, with eigenvector $\underline{a}$.

$\therefore$ Sol^n to (2) is

$$\begin{pmatrix} \tilde{\Theta} \\ \tilde{\Psi} \end{pmatrix} = \alpha \underline{a} e^{\lambda_1 \tau} + \beta \underline{b} e^{\lambda_2 \tau} \qquad (3)$$

where λ_1 (λ_2) is evalue of J with evector $\underline{a}$ ($\underline{b}$).

α, β determined by initial cond^{ns}.

Case 1. $\lambda_1 > \lambda_2 > 0$.

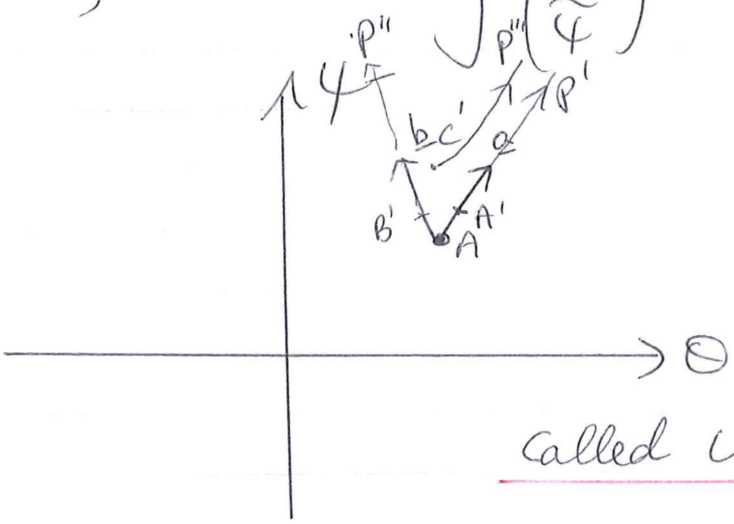
Let init. cond^n $\begin{pmatrix} \tilde{\Theta} \\ \tilde{\Psi} \end{pmatrix}_{\tau=0}$ be the vector $\underline{c}$.

Since $\underline{a}$, $\underline{b}$ are lin indep, they form a basis vector pair. $\therefore$ $\underline{c} = c_1 \underline{a} + c_2 \underline{b}$ for some consts c_1, c_2. $\therefore$ $\begin{pmatrix} \tilde{\Theta} \\ \tilde{\Psi} \end{pmatrix} = c_1 \underline{a} e^{\lambda_1 \tau} + c_2 \underline{b} e^{\lambda_2 \tau}$.

Suppose $c_2 = 0$, then $\begin{pmatrix}\hat{\Theta}\\ \tilde{\Psi}\end{pmatrix} = c_1 \underline{a} e^{\lambda_1 \tau}$

Likewise, if $c_1 = 0$, then $\begin{pmatrix}\hat{\Theta}\\ \tilde{\Psi}\end{pmatrix} = c_2 \underline{b} e^{\lambda_2 \tau}$

In general, c_1 & c_2 both non-zero, but, as $\lambda_1 > \lambda_2$, eventually $\begin{pmatrix}\hat{\Theta}\\ \tilde{\Psi}\end{pmatrix} \sim c_1 \underline{a} e^{\lambda_1 \tau}$



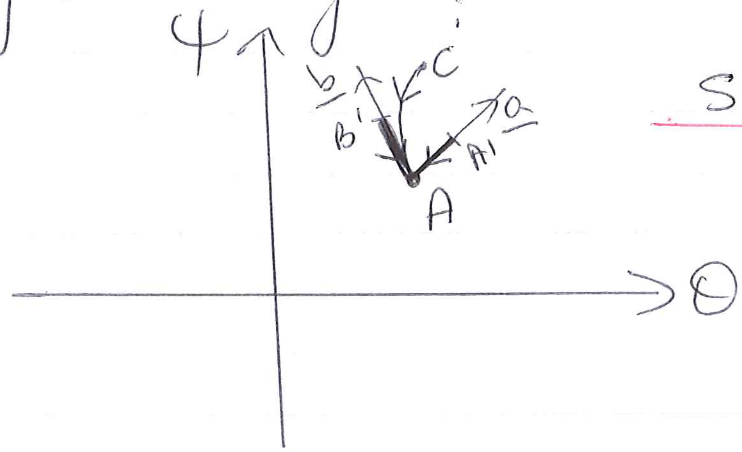
A is (Θ^*, Ψ^*)

Called unstable NODE

if I start at A', I will follow A'P'
- - - - B' - - - - B'P''
- - - - C' - - - - C'P'''

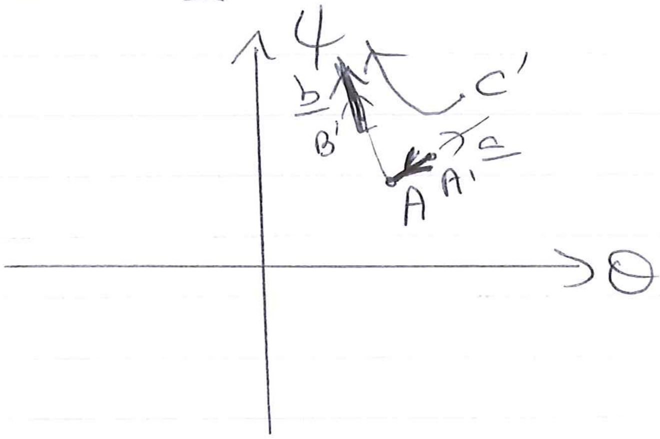
Case 2 Suppose $\lambda_1 < \lambda_2 < 0$.

Apply same arguments:



STABLE NODE

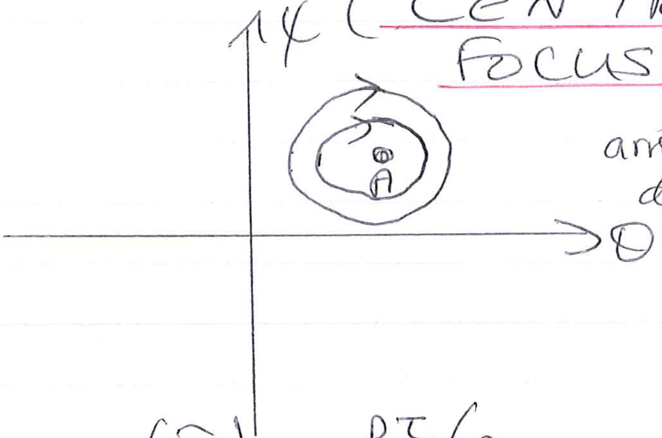
Case 3 $\lambda_1 < 0 < \lambda_2$



Case 4 $\lambda_1, \lambda_2 = p \pm iq$ (complex conjugates).

(i) $p = 0 \Rightarrow \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \end{pmatrix} = P_1 \cos q\tau + P_2 \sin q\tau$

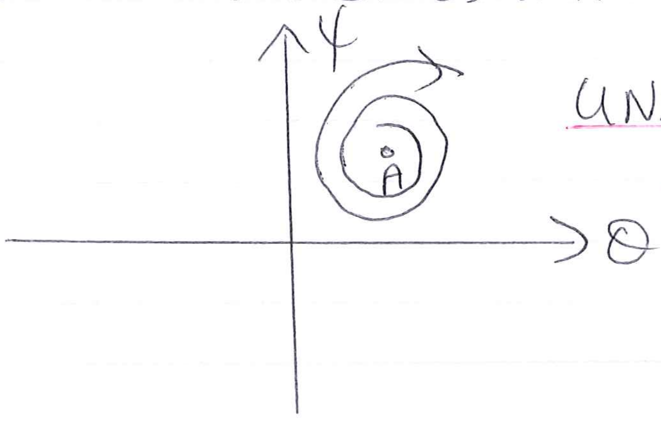
$\Rightarrow$ closed curve (CENTRE or FOCUS)



arrow direction depends on $\mathring{\theta}, \mathring{\psi}$ etc.

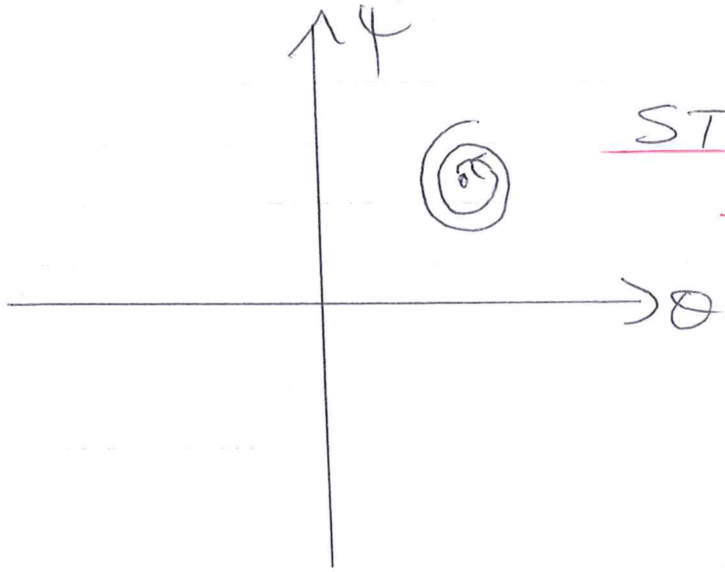
(ii) $p > 0 \Rightarrow \begin{pmatrix} \tilde{\theta} \\ \tilde{\psi} \end{pmatrix} = e^{p\tau}\left(P_1 \cos q\tau + P_2 \sin q\tau\right)$

i.e., amplitude increases with $\tau \uparrow$



UNSTABLE SPIRAL

(iii) $p<0 \Rightarrow$ amplitude decreases with $\tau\uparrow$



STABLE SPIRAL