

MSc Supplementary Course: Mathematical Methods I

Andreas Münch
based on notes by Derek E. Moulton

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Health warning:

The following lecture notes are meant as a rough guide to the lectures. They are not meant to replace the lectures. You should expect that some material in these notes will not be covered in class and that extra material will be covered during the lectures (especially longer proofs, examples, and applications). Nevertheless, I will try to follow the notation and the overall structure of the notes as much as possible. Also, these notes may be updated during the course of the term. In particular, please alert me if you catch any typos or errors. I will notify you if I upload an updated version.

1 Introduction

In this course, we will explore various techniques for solving differential equations, building on basic techniques for solving Differential Equations. A primary concern will be finding ways to solve and understand inhomogeneous linear boundary value problems (BVPs), that is an ordinary differential equation (ODE)

$$Lu(x) = f(x), \quad a < x < b \quad (1)$$

where L is a linear differential operator of the form

$$Lu = a_n u^{(n)}(x) + a_{n-1} u^{(n-1)}(x) + \cdots + a_1 u'(x) + a_0 u(x) \quad (2)$$

and the function $f(x)$ on the right hand side (RHS) is a *forcing function* in the system. Along with equation (1) are boundary conditions at $x = a$ and $x = b$.

Some questions we will consider:

1. How do we solve the system for an arbitrary function $f(x)$?
2. Is there always a solution? If so, is it unique?
3. What is the effect of the boundary conditions?
4. Can we solve if the $a_k = a_k(x)$ are functions of x ?

1.1 Motivating example: The diffusion equation

The diffusion equation models many phenomena. Let $u = u(x, t)$ denote a quantity that depends on one spatial variable x and time $t > 0$. This could be a chemical concentration, population, mass, energy, etc. The diffusion equation can be derived in a very intuitive way: we imagine a segment of space $[a, b]$. In this segment the total amount of “stuff” is

$$\int_a^b u(x, t) A \, dx \quad (3)$$

where A is the cross-sectional area. Now, we simply say that the rate of change of “stuff” in the segment is equal to the amount that leaves/enters through the ends plus any stuff that is added/taken away by the external world. The amount leaving the ends is the flux, denoted ϕ , and let $f(x, t)$ be a local source function – this is the rate u is created or destroyed at x at time t ; note it could depend on u itself.

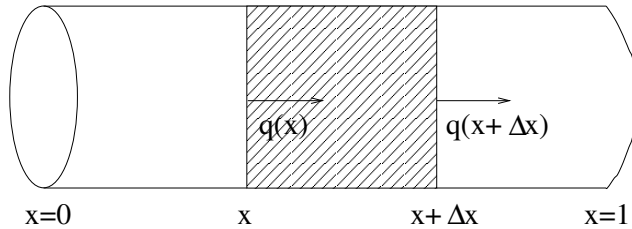


Figure 1: 1D diffusion

The balance is

$$\frac{d}{dt} \int_a^b u(x, t) \, dx = \phi(b, t) - \phi(a, t) + \int_a^b f(x, t, u) \, dx \quad (4)$$

By recognising

$$\phi(b, t) - \phi(a, t) = \int_a^b \phi_x(x, t) dx$$

we obtain

$$\int_a^b u_t + \phi_x - f dx = 0 \quad (5)$$

But this should hold for any segment, and thus the integrand must vanish:

$$u_t + \phi_x = f \quad (6)$$

To complete the system, the flux needs to be related to the quantity u . The simplest way is through Fick's Law, which states

$$\phi(x, t) = -Du_x(x, t) \quad (7)$$

where D is the diffusion constant. Combining (6) and (7), we obtain the diffusion equation

$$u_t - Du_{xx} = f. \quad (8)$$

The classic heat equation is the case $f = 0$. Many interesting and physically relevant situations are modelled with non-zero f . For example, (8) is a popular model for population dynamics, where f is used to capture growth and other interactions of the population. To seek stationary solutions, we let $u_t = 0$ in (8), and we obtain an equation of the type (1), (2) albeit a very simple one.

2 Eigenfunction methods

Our first approach to solving linear inhomogeneous BVP's is through an eigenfunction expansion. The idea is to exploit the linearity of the operator by constructing a solution as a *superposition* of a (generally infinite) set of functions $\{y_i(x)\}$. In particular, the y_i will be the functions satisfying

$$Ly_i(x) = \lambda_i y_i(x), \quad (9)$$

along with homogeneous boundary conditions. Here y_i is an *eigenfunction* with corresponding *eigenvalue* λ_i . This is analogous to the linear algebra eigenproblem

$$\mathbf{A}\vec{x}_i = \lambda_i \vec{x}_i \quad (10)$$

where \mathbf{A} is a matrix and \vec{x}_i an eigenvector with eigenvalue λ_i .

2.1 Function spaces

In the same way as linear algebra utilises vector spaces, with linear differential operators we shall think of function spaces. Consider the infinite dimensional space of all reasonably well-behaved functions on the interval $a \leq x \leq b$.

Similar to a vector space, we can introduce a set of linearly independent basis functions $y_n(x)$, $n = 1, 2, \dots \infty$ such that any ‘reasonable’ function $f(x)$ can be written as a linear combination of these functions:

$$f(x) = \sum_{n=0}^{\infty} c_n y_n(x). \quad (11)$$

You have encountered this idea before with Fourier Series, where the basis functions are sines and cosines; this is merely a generalisation. Hence it should be clear that we can have different sets of basis functions.

We also define the **inner product**

$$\langle u, v \rangle = \int_a^b u(x) \overline{v(x)} dx. \quad (12)$$

Here the overbar denotes complex conjugate. In this course, we will rarely be concerned with complex valued functions. If it is clear that we are dealing with real functions, we may drop the overbar for simplicity.

2.1.1 Weighting function

In some instances, the eigenvalue problem and the inner product definition might include a given weighting function $\rho(x)$, required to be real and not change sign on $a \leq x \leq b$. In this case, the relations become

$$Ly_i(x) = \lambda_i \rho(x) y_i(x) \quad (13)$$

and

$$\langle u, v \rangle = \int_a^b \rho(x) u(x) \overline{v(x)} dx. \quad (14)$$

2.2 Adjoint

We also require the notion of the *adjoint* of an operator. For operator L with homogenous BC, the adjoint problem (L^* BC*) is defined by the inner product relation

$$\langle Ly, w \rangle = \langle y, L^* w \rangle. \quad (15)$$

To determine the adjoint, one needs to move the derivatives of the operator from y to w , and define adjoint boundary conditions so that all boundary terms vanish.

Example

Let $Ly = \frac{d^2y}{dx^2}$ with $a \leq x \leq b$, $y(a) = 0$ and $y'(b) - 3y(b) = 0$. We wish to find L^*w , such that

$$\int_a^b (w)(y'')dx = \int_a^b (y)(L^*w)dx$$

To do this, we need to shift the derivatives from y to w using integration by parts:

$$\begin{aligned} \int_a^b wy''dx &= wy'|_a^b - \int_a^b w'y'dx \\ &= wy' - w'y|_a^b + \int_a^b yw''dx \end{aligned}$$

The integral gives the formal part so:

$$L^*w = \frac{d^2w}{dx^2}.$$

The inner product only includes integral terms, so *the boundary terms must vanish*, which will define boundary conditions on w , i.e. this defines BC^* . Here, we require

$$y(b)y'(b) - w'(b)y(b) - w(a)y'(a) + w'(a)y(a) = 0.$$

Using the BC's $y'(b) = 3y(b)$ and $y(a) = 0$, gives:

$$0 = y(b)(3w(b) - w'(b)) - w(a)y'(a) + \underbrace{w'(a)y(a)}_{=0}$$

As these terms need to vanish for all values of $y(b)$ and $y'(a)$, we can infer two boundary conditions on w :

- $y(b)$: $3w(b) - w'(b) = 0$
- $y'(a)$: $w(a) = 0$

If $L = L^*$ and $BC = BC^*$ then the problem is *self-adjoint*. If $L = L^*$ but $BC \neq BC^*$ we still call the operator self-adjoint. (Some books use the terminology *formally self-adjoint* if $L = L^*$ and *fully self-adjoint* if both $L = L^*$ and $BC = BC^*$).

2.2.1 Eigenfunction Properties

The main idea in solving the BVP is to construct a solution as a linear combination of eigenfunctions. There are two fundamental properties of eigenfunctions that will be vital to this approach.

1. **Eigenfunctions of the adjoint problem have the same eigenvalues as the original problem**

That is,

$$Ly = \lambda y \Rightarrow \exists w \ni L^*w = \lambda w.$$

2. **Eigenfunctions corresponding to different eigenvalues are orthogonal**

That is, if $Ly_j = \lambda_j y_j$ (so $L^*w_j = \lambda_j w_j$) and $Ly_k = \lambda_k y_k$ ($L^*w_k = \lambda_k w_k$), then for $\lambda_j \neq \lambda_k$, $\langle y_j, w_k \rangle = 0$.

Proof

$$\begin{aligned} \lambda_j \langle y_j, w_k \rangle &= \langle \lambda_j y_j, w_k \rangle \\ &= \langle Ly_j, w_k \rangle \\ &= \langle y_j, L^*w_k \rangle \\ &= \langle y_j, \lambda_k w_k \rangle \\ &= \lambda_k \langle y_j, w_k \rangle. \end{aligned}$$

But $\lambda_j \neq \lambda_k$ so $\langle y_j, w_k \rangle = 0$. (The proof is exactly as for matrix problems.)

2.3 Inhomogeneous solution process

We are now in a position to outline the construction of solution to the BVP

$$Ly = f(x)$$

with linear, homogeneous, separated boundary conditions, denoted $BC_1(a) = 0$, $BC_2(b) = 0$.

Step 1: Solve the eigenvalue problem

$$Ly = \lambda y, \quad BC_1(a) = 0, \quad BC_2(b) = 0$$

to obtain the eigenvalue-eigenfunction pairs (λ_j, y_j) .

Step 2: Solve the adjoint eigenvalue problem

$$L^*w = \lambda w, \quad BC_1^*(a) = 0, \quad BC_2^*(b) = 0$$

to obtain (λ_j, w_j) .

Step 3: Assume a solution to the full system $Ly = f(x)$ of the form

$$y = \sum_i c_i y_i(x).$$

To determine the coefficients c_i , start from $Ly = f$ and take an inner product with w_k :

$$\begin{aligned} Ly &= f(x) \\ \Rightarrow \langle Ly, w_k \rangle &= \langle f, w_k \rangle \\ \Rightarrow \langle y, L^* w_k \rangle &= \langle f, w_k \rangle \\ \Rightarrow \langle y, \lambda_k w_k \rangle &= \langle f, w_k \rangle \\ \Rightarrow \lambda_k \langle \sum_i c_i y_i, w_k \rangle &= \langle f, w_k \rangle \\ \Rightarrow \lambda_k c_k \langle y_k, w_k \rangle &= \langle f, w_k \rangle \end{aligned} \tag{16}$$

We can solve the last equality for the c_k , and we are done! Note that in the last step we have used the orthogonality property $\langle y_j, w_k \rangle = 0$, $j \neq k$.

2.4 Some simple solutions

Note that this construction requires that we are able to determine the eigenvalues/eigenfunctions in the first place. This is by no means guaranteed. But it will be useful to recall some simple cases, solvable using techniques you've learned before:

- Constant coefficients

$$Ly \equiv ay'' + by' + cy = \lambda y$$

Try $y = e^{mx}$, then:

$$am^2 + bm + (c - \lambda) = 0$$

Then:

1. Find roots m_i of the quadratic.
2. The general solution is: $y = A_1 e^{m_1 x} + A_2 e^{m_2 x}$. But note there are 3 unknowns: A_1 , A_2 , and λ , while for a second order equation there will only be two BC's.
3. Apply first BC to relate A_1 and A_2 .
4. Apply second BC to determine values for λ .

• Cauchy-Euler

$$Ly \equiv ax^2 y'' + bxy' + cy = \lambda y$$

Try $y = x^m$, then:

$$am(m-1) + bm + (c - \lambda) = 0$$

Then $y = A_1 x^{m_1} + A_2 x^{m_2}$, and repeat the steps above.

2.5 A note on boundary conditions

In the above construction we assumed homogeneous boundary conditions. In the general case of an inhomogeneous system with inhomogeneous boundary conditions,

$$\begin{aligned} Lu &= f(x) \\ B_i u &= \gamma_i \end{aligned} \tag{17}$$

a useful technique is to split the system in two, i.e. solve both

$$Lu_1 = f(x), \quad B_i u_1 = 0 \tag{18}$$

and

$$Lu_2 = 0, \quad B_i u_2 = \gamma_i. \tag{19}$$

Here, solving for $u_1(x)$ has the difficulty of the forcing function but with zero BC's while the other equation is homogeneous but has the non-zero BC's. Due to linearity, it is easy to see that $u(x) = u_1(x) + u_2(x)$ solves the full system (17).

This decomposition can always be performed¹ and since solving (19) tends to be an easier matter (for linear systems!), it is safe for us to primarily focus on the technique of solving the system (18), i.e. homogeneous boundary conditions.

¹As we shall see in Section ??, it requires caution if there is a zero eigenvalue $\lambda = 0$.

For completeness it is worth noting that one *can* solve BVPs with inhomogeneous BC using an eigenfunction expansion and without needing a decomposition. The keys are:

1. The eigenfunctions are *always* determined using homogeneous boundary conditions. Thus, eigenfunctions won't change whether you "decompose" or not. The difference comes in:
2. In going from Line 2 to 3 of (16), care must be taken in the integration by parts, as boundary terms will generally still be present. (Can you see why?) These extra boundary terms then carry through to the formula for the c_k .

2.6 Example

Let $y'' = f(x)$ with $0 \leq x \leq 1$, $y(0) = \alpha$ and $y(1) = \beta$. Then:

BC's Incorporated Solution Route

1. Solve $y'' = \lambda y$, with $y(0) = 0$ and $y(1) = 0$.

We get $y_k(x) = \sin(k\pi x)$ and $\lambda_k = -k^2\pi^2$ with $k = 1, 2, 3, \dots$

The problem is self-adjoint (show this as an exercise), so $w_k = y_k = \sin(k\pi x)$ and $w_k'' = -\lambda_k w_k$.

- 2.

$$\begin{aligned}
 y'' &= f(x) \\
 \int_0^1 w_k y'' dx &= \int_0^1 w_k f dx \\
 \Rightarrow (y' w_k - y w_k')|_0^1 + \int_0^1 w_k'' y dx &= \int_0^1 w_k f dx \\
 \Rightarrow (y' w_k - y w_k')|_0^1 + \lambda_k \int_0^1 w_k y dx &= \int_0^1 w_k f dx \\
 \Rightarrow (y' w_k - y w_k')|_0^1 + \lambda_k c_k \int_0^1 w_k y_k dx &= \int_0^1 w_k f dx \\
 \Rightarrow (y' w_k - y w_k')|_0^1 - k^2 \pi^2 c_k \int_0^1 \sin^2(k\pi x) dx &= \int_0^1 w_k f dx
 \end{aligned}$$

3. Now $\int_0^1 \sin^2(k\pi x) dx = 1/2$, and $w_k = \sin(k\pi x)$, hence

$$\begin{aligned} y'w_k - yw'_k|_0^1 &= -k\pi \cos(k\pi)y(1) + k\pi \cos(0)y(0) \\ \Rightarrow -\beta k\pi(-1)^k + \alpha k\pi - \frac{1}{2}k^2\pi^2 c_k &= \int_0^1 f(x) \sin k\pi x dx \end{aligned}$$

Solving for c_k gives us $y(x)$ as a Fourier series.

Decomposed Solution Route

1. Solve two systems separately:

$$\begin{aligned} y'' &= f(x), & y(0) &= y(1) = 0 \\ u'' &= 0, & u(0) &= \alpha, u(1) = \beta \end{aligned}$$

2. To solve for y , since BC=0 we can jump straight to the formula

$$c_k = -\frac{\langle f, w_k \rangle}{\lambda_k \langle y_k, w_k \rangle} = -\frac{2 \int_0^1 f(x) \sin(k\pi x) dx}{k^2 \pi^2}.$$

3. The solution for u is easily obtained as

$$u = (\beta - \alpha)x + \alpha$$

4. The full solution is $y(x) + u(x)$.

Although they look different, both approaches give the same solution. Either way, we see that self-adjoint problems are great: they are less work since the w_k 's are the same as the y_k 's.

2.7 Connection with linear algebra

There are direct parallels between linear algebra and linear BVPs:

Linear algebra		Linear BVP
vector $\vec{v} \in \mathbb{R}^n$	\longleftrightarrow	function $y(x)$ for $a \leq x \leq b$
$\underbrace{\vec{v} \cdot \vec{w} = \sum_{k=1}^n v_k w_k}_{\text{dot product}}$	\longleftrightarrow	$\underbrace{\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx}_{\text{inner product}}$
$\underbrace{\ \vec{v}\ ^2 = \vec{v} \cdot \vec{v} \geq 0}_{\text{norm}}$	\longleftrightarrow	$\underbrace{\ f\ ^2 = \langle f, f \rangle \geq 0}_{\text{norm}}$
\perp vector $\vec{v} \cdot \vec{w} = 0$	\longleftrightarrow	orthogonal function $\langle f, g \rangle = 0$

$$\text{Matrix } A \longleftrightarrow \text{Linear Differential Operator } L$$

Given a vector \vec{v} , then the product $A\vec{v}$ is a new vector. Similarly, given a function $y(x)$,

$$Ly = a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy$$

evaluates to a new function on $a \leq x \leq b$.

In linear algebra, a common problem is to solve the equation

$$A\vec{v} = \vec{b}$$

for \vec{v} , given matrix A and vector \vec{b} . Compare that to our general task of solving $Ly = f$ for y , given operator L and RHS f .

Eigenvalue problems

Linear algebra		Linear BVP
$A\vec{v} = \lambda \vec{v}$	\longleftrightarrow	$Ly = \lambda y$

How many eigenvalues?

<u>Linear algebra</u>	<u>Linear BVP</u>
A is $n \times n$	L is order n
Solve $ A - \lambda I = 0$	
$\Rightarrow n$ eigenvalues	∞ eigenvalues

Adjoint

	<u>Linear algebra</u>	<u>Linear BVP</u>
	$A \rightarrow A^T$	$L \rightarrow L^*$ BC's \rightarrow BC*'s
Self adjoint if	$A = A^T$	$L = L^*, \text{ BC} = \text{BC}^*$

A self-adjoint matrix is called Hermitian. A self-adjoint linear differential operator is also referred to as Hermitian. We next look at a particular class of Hermitian operator – Sturm-Liouville operators – that occurs quite commonly and has very useful properties.

2.8 Sturm-Liouville theory

Sturm-Liouville (SL) theory of second order concerns self-adjoint operators of the form:

$$Ly = \lambda r(x)y$$

where $r(x) \geq 0$ is a *weighting function*, and the operator L is of the form

$$Ly = -\frac{d}{dx} \left(p(x) \frac{dy}{dx} \right) + q(x)y, \quad a \leq x \leq b \quad (20)$$

The functions p, q , and r are all assumed to be real. It is easy to check that the operator is formally self-adjoint. It is fully self-adjoint if the boundary conditions take the separated form

$$\begin{aligned} \alpha_1 y(a) + \alpha_2 y'(a) &= 0 \\ \alpha_3 y(b) + \alpha_4 y'(b) &= 0. \end{aligned}$$

Observe also that if $p(a) = p(b) = 0$, then $\langle Ly, w \rangle = \langle y, Lw \rangle$ irrespective of boundary conditions. This defines the so-called natural interval $[a, b]$ for the problem.

2.8.1 Inhomogeneous SL problems

Since a SL operator is self-adjoint, the eigenfunction expansion process is quite straightforward. Consider

$$Ly = f(x)$$

with homogeneous BC's. The system can be solved with an eigenfunction expansion in the same manner as in Section 2.3:

$$\begin{aligned} Ly &= f(x) \\ \Rightarrow \langle Ly, y_k \rangle &= \langle f, y_k \rangle \\ \Rightarrow \langle y, Ly_k \rangle &= \langle f, y_k \rangle \quad (\text{since } L^* = L, w_k = y_k) \\ \Rightarrow \langle y, \lambda_k r y_k \rangle &= \langle f, y_k \rangle \\ \Rightarrow \lambda_k c_k \langle y_k, r y_k \rangle &= \langle f, y_k \rangle. \end{aligned} \tag{21}$$

Thus we obtain the formula

$$c_k = \frac{\langle f, y_k \rangle}{\lambda_k \langle y_k, r y_k \rangle} \tag{22}$$

and the full solution is given by

$$y = \sum_k c_k y_k.$$

2.8.2 Transforming an operator to SL form

Many problems encountered in physical systems are Sturm-Liouville. In fact, though, any operator

$$Ly \equiv a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x)$$

with $a_2(x) \neq 0$ in the interval can be converted to a SL operator.

To *transform* to a self-adjoint SL operator, multiply by an integrating factor function $\mu(x)$:

$$\mu a_2(x)y''(x) + \mu a_1(x)y'(x) + \mu a_0(x)y$$

We then choose μ so that the first and second derivatives collapse, i.e. so it can be expressed in the form

$$-\frac{d}{dx}(py') + qy$$

Suppose we are considering the problem

$$Ly = f(x)$$

where L is not Sturm-Liouville. We could solve following the approach in Eq'n (16); alternatively we could convert to Sturm-Liouville first, and then proceed using the nice properties of a self-adjoint operator. So, is the problem self-adjoint or isn't it?? The key observation is that we are no longer solving the same problem. We have transformed to a new operator

$$\hat{L}y = -\frac{d}{dx}(py') + qy$$

which does not satisfy the same equation as the original, that is $Ly = f$ while $\hat{L}y = \mu f$. They are both valid, and must ultimately lead to the same answer in the end.

2.8.3 Further properties

Orthogonality.

Due to the presence of the weighting function, the *orthogonality* relation is

$$\int_a^b y_k(x)y_j(x)r(x)dx = 0. \quad (23)$$

Eigenvalues.

The functions p, q, r are real, so $\bar{L} = L$. Thus, taking the conjugate of both sides of $Ly_k = \lambda_k y_k$ gives

$$\begin{aligned} L \bar{y}_k &= \bar{\lambda}_k r \bar{y}_k \\ \Rightarrow \langle y_k, L \bar{y}_k \rangle &= \bar{\lambda}_k \langle y_k, r \bar{y}_k \rangle \\ \text{but } \langle y_k, L \bar{y}_k \rangle &= \langle Ly_k, \bar{y}_k \rangle = \lambda_k \langle r y_k, \bar{y}_k \rangle = \lambda_k \langle y_k, r \bar{y}_k \rangle \\ \Rightarrow \bar{\lambda}_k &= \lambda_k \end{aligned} \quad (24)$$

Thus, all eigenvalues are real.

Moreover, if $a \leq x \leq b$ is a finite domain, then λ 's are discrete and countable:

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_k < \cdots$$

with $\lim_{k \rightarrow \infty} \lambda_k = \infty$.

Eigenfunctions.

The $\{y_k\}$ are a *complete set*, that is all $h(x)$ with $\int h^2 r dx < \infty$ can be expanded as

$$h(x) = \sum c_k y_k(x).$$

Take an inner product with $r(x)y_j(x)$:

$$\begin{aligned} \langle ry_j, h \rangle &= \langle ry_j, \sum c_k y_k \rangle = \sum c_k \langle ry_j, y_k \rangle = c_j \langle ry_j, y_j \rangle \\ \Rightarrow c_j &= \frac{\int_a^b h(x) y_j(x) r(x) dx}{\int_a^b y_j^2(x) r(x) dx} \end{aligned}$$

Note: I've used $h(x)$ to make clear that we're not talking about the solution to the BVP, rather we are expanding any function that is suitably bounded on the same domain.

2.8.4 Other tidbits

Regular Sturm-Liouville Problems. If the system satisfies all of the above and the additional conditions

- $p(x) > 0$ and $r(x) > 0$ on $a \leq x \leq b$.
- $q(x) \geq 0$ on $a \leq x \leq b$.
- BCs have $\alpha_1 \alpha_2 \leq 0$ and $\alpha_3 \alpha_4 \geq 0$,

then all $\boxed{\lambda_k \geq 0}$

Proof: Using $\langle y_k, Ly_k - \lambda_k r y_k \rangle = 0$,

$$\begin{aligned} - \int_a^b y(py')' dx + \int_a^b yqy dx - \int_a^b y\lambda r y dx &= 0 \\ - \int_a^b y(py')' dx + \int_a^b qy^2 dx - \lambda \int_a^b ry^2 dx &= 0 \\ -pyy' \Big|_a^b + \int_a^b p(y')^2 dx + \int_a^b qy^2 dx - \lambda \int_a^b ry^2 dx &= 0 \\ \lambda = \left[\int_a^b p(y')^2 dx + \int_a^b qy^2 dx - p y y' \Big|_a^b \right] / \int_a^b ry^2 dx &\geq 0 \end{aligned}$$

As a side note, the **Rayleigh quotient**, $R[y] = \langle y, Ly \rangle / \langle y, ry \rangle$, is used extensively in analysis.

Oscillation theorem [Simplest version]: The k^{th} eigenfunction will have k zeroes on $a < x < b$ ($k = 0, 1, 2, \dots$).

Monotonicity theorem: Comparing two SL problems, SL and $\widetilde{\text{SL}}$, with the same boundary conditions, the eigenvalues will satisfy $\tilde{\lambda}_k > \lambda_k$ if

$$\tilde{p}(x) \geq p(x) \quad \text{and} \quad \tilde{q}(x) \geq q(x) \quad \text{and} \quad \tilde{r}(x) \leq r(x) \quad \text{and} \quad (\tilde{a}, \tilde{b}) \subseteq (a, b)$$

and the strict equality (or strict inclusion) holds in at least one of these cases.

3 Green's function

In this section we will devise an alternative approach to viewing and solving linear BVP, using the so-called Green's function.

3.1 Form of the eigenfunction expansion solution

Consider the form of the final solution obtained through the eigenfunction expansion approach. Taking (16) one step further, we have

$$y(x) = \sum_{k=1}^{\infty} \frac{\langle f, w_k \rangle}{\lambda_k \langle y_k, w_k \rangle} y_k(x)$$

(Of course, this requires all $\lambda_k \neq 0$. The case of zero eigenvalue has two subcases: One where $\langle f, w_k \rangle \neq 0$ and one where this inner product is zero. In the former sub case, the boundary value problem has no solution, in the latter, the solution is not unique as you can add any multiple of y_k (the eigenfunction that belongs to the zero eigenvalue) to the solution. This observation is directly linked to the Fredholm Alternative, which will be discussed in “Further Mathematical Methods” in HT.)

Let $n_k = \langle y_k, w_k \rangle$ (normalisation), then:

$$\begin{aligned} y(x) &= \sum_{k=1}^{\infty} \frac{1}{\lambda_k n_k} \left(\int_a^b f(t) w_k(t) dt \right) y_k(x) \\ &= \int_a^b \left(\sum_{k=1}^{\infty} \frac{1}{\lambda_k n_k} w_k(t) y_k(x) \right) f(t) dt \\ &= \int_a^b g(x, t) f(t) dt \end{aligned}$$

where

$$g(x, t) = \sum_{k=1}^{\infty} \frac{w_k(t)y_k(x)}{\lambda_k n_k}. \quad (25)$$

Thus, we have constructed a solution to $Ly = f$ in the form

$$y(x) = \int_a^b g(x, t)f(t) dt. \quad (26)$$

The function $g(x, t)$ is called the Green's function (GF), and the form (25) is an eigenfunction expansion of $g(x, t)$.

Of course, if we knew the Green's function, we would have the solution without any need for the expansion, i.e. no need for the eigenfunctions. The goal in this section is to understand the properties of the GF and how to construct it.

Side note: Observe that if $L = L^*$, then $w_k = y_k$ and:

$$g(x, t) = \sum \frac{1}{\lambda_k n_k} y_k(t)y_k(x)$$

In this case $g(x, t) = g(t, x)$, and we have the important connection between a self-adjoint operator and a symmetric Green's function.

3.2 Inverse of differential operator

A nice way to think of the Green's function is in terms of inverting the differential operator. Think about the familiar equation $\mathbf{A}\vec{x} = \vec{b}$ from linear algebra, to be solved for the unknown vector \vec{x} . The solution is given by

$$\vec{x} = \mathbf{A}^{-1}\vec{b},$$

i.e. we find the solution by multiplying the inverse of the linear operator (matrix) by the inhomogeneous term. Once you know the inverse operator, you can solve the problem for any given vector \vec{b} . In the context of BVP's, L is a differential operator, so it stands to reason that the inverse operator involve integration, hence the form (26). Constructing the Green's function is analogous to finding the inverse of the matrix, once we have g we can write down the solution (26) for any forcing function $f(x)$.

3.2.1 An example

There are numerous ways to construct a Green's function. We've already seen one: the eigenfunction expansion. Another way that you've probably seen before is via variation of parameters². This approach gives the Green's function in a piecewise form.

Let's take a simple example and look at the behaviour. Consider the BVP:

$$\begin{aligned} Ly \equiv -y'' &= f(x), 0 < x < 1 \\ y(0) &= y(1) = 0 \end{aligned} \quad (27)$$

The GF via variation of parameters is given by

$$g(x, \xi) = \begin{cases} (1 - \xi)x & 0 < x < \xi \\ (1 - x)\xi & \xi < x < 1. \end{cases} \quad (28)$$

The following properties are easily checked:

- The GF satisfies $Lg = 0$ if $x \neq \xi$ ³
- $g(x, \xi)$ satisfies the boundary conditions *as a function of x* .
- g is continuous on the whole interval $[0, 1]$
- g is differentiable everywhere *except at $x = \xi$* , where it suffers a jump in the derivative.

These properties are in fact always true of the GF of a second order linear operator.⁴ To make sense of this, and to build some physical intuition, we shall need the notion of the delta function.

3.3 Green's function via delta function

To fix the context, consider stationary heat conduction in a rod:

$$-y''(x) = f(x) \quad 0 < x < 1 \quad (29)$$

$$y(0) = 0, \quad y(1) = 0. \quad (30)$$

where $y(x)$ is the temperature field and $f(x)$ is a given heat source density.

²You will not be tested on this method, but if you would like to review the approach you might check out the textbook *Elementary Differential Equations and Boundary Value Problems*, by Boyce and DiPrima.

³Here by L we mean the operator acting on the x variable, i.e. derivatives are with respect to x – this is sometimes written $L_x g(x, \xi)$ to clarify.

⁴Note, however, that the function $y(x)$ satisfying $Ly = f$ is continuously differentiable assuming continuously differentiable f , meaning that the integration with $f(x)$ smooths out the discontinuity in g .

3.3.1 Delta function

The function $f(x)$ describes any heat added or removed from the system by the outside world. As a simple scenario, consider a *point heat source*, say located at the middle of the rod. Physically, this would correspond to applying heat at a single point only. How would we describe such a situation mathematically? What should we use for the function $f(x)$?

The notion of a point source is described by the “delta function” δ , characterised by properties

$$\delta(x) = 0 \quad \forall x \neq 0, \quad \int_{-\infty}^{\infty} \delta(x) dx = 1. \quad (31)$$

The first property captures the notion of a point function. The second property constrains the area under the curve (which you might think of as infinitely thin and infinitely high). This is an idealized point source at $x = 0$, a point source at $x = a$ would be given by $\delta(x - a)$.

The problem is that no classical function satisfies (31) (think: any function that is non-zero only at a point is either not integrable or integrates to zero).

3.3.2 Approximating the delta function

One way around this is to replace δ by an approximating sequence of increasingly narrower functions with normalized area, i.e. $f_n(x)$ where

$$\int_{-\infty}^{\infty} f_n(x) dx = 1 \quad \forall n, \quad \lim_{n \rightarrow \infty} f_n(x) = 0 \quad \forall x \neq 0.$$

Example: “hat” functions

$$f_n(x) = \begin{cases} 0 & \text{for } |x| > 1/n \\ n/2 & \text{for } |x| \leq 1/n \end{cases} \quad (32)$$

You can verify the $f_n(x)$ approach $\delta(x)$ as $n \rightarrow \infty$.

3.3.3 Properties of delta function

We have defined δ by (31). We can use the approximating functions to obtain further properties.

Sifting property. What happens when δ is integrated against another function?

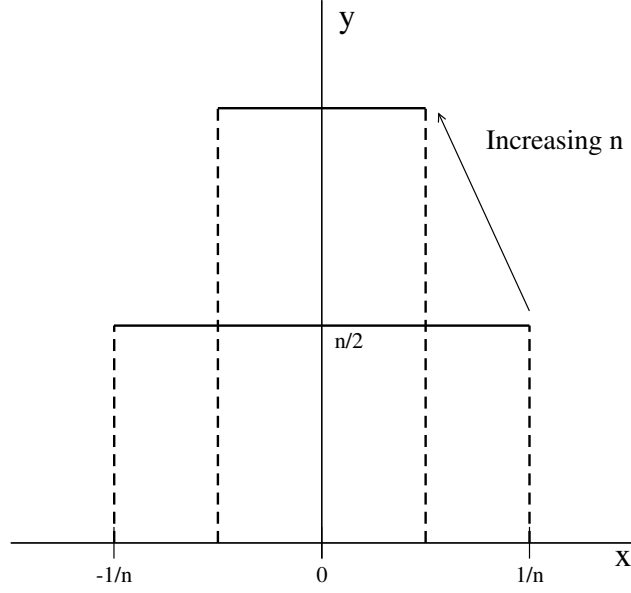


Figure 2: Hat functions, see eqn. (32)

Let $f(x)$ be a continuous function, and $F(x) = \int^x f(s)ds$ its antiderivative. Now consider approximating sequences:

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = \lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x-a)f(x)dx,$$

and if f_n are the hat functions (32),

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \int_{a-1/n}^{a+1/n} \frac{n}{2} f(x)dx = \lim_{n \rightarrow \infty} \frac{F(a + (1/n)) - F(a - (1/n))}{2/n} \\ &= \lim_{s \rightarrow 0} \frac{F(a + s) - F(a - s)}{2s} = F'(a) = f(a). \end{aligned}$$

Thus, we have

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a) \quad \text{if } f \text{ is continuous at } a. \quad (33)$$

In particular,

$$\int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0) \quad \text{if } f \text{ is continuous at } x = 0. \quad (34)$$

Thus, the delta function can be seen to sift out the value of a function at a particular point.

Antiderivative of $\delta(x)$. The antiderivative of the delta function is the so-called Heaviside function,

$$\int_{-\infty}^x \delta(s) ds = H(x) \equiv \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases} \quad (35)$$

Note that (35) follows by integrating the sequence of approximating functions and showing that the limit is the Heaviside function. That is, if $H_n(x) = \int_{-\infty}^x f_n(s) ds$, then $\lim_{n \rightarrow \infty} H_n(x) = H(x)$. (We leave this detail as an exercise!)

3.3.4 Point heat source

Let's return to the heat conduction BVP with a point heat source of unit strength at the centre of the rod:

$$-y''(x) = \delta(x - 1/2), \quad 0 < x < 1 \quad (36)$$

$$y(0) = y(1) = 0. \quad (37)$$

Since $\delta(x - 1/2) = 0 \quad \forall x \neq 1/2$, this implies

$$-y''(x) = 0, \quad 0 < x < 1/2, \quad 1/2 < x < 1. \quad (38)$$

We can easily solve (38) in each of the two separate domains $[0, 1/2)$ and $(1/2, 1]$ and then apply the BC (37). But be careful: there are two constants of integration for each domain, meaning *four* unknown constants total, and only *two* boundary conditions.

As you might expect (since $\delta(x - 1/2)$ has vanished from (38)), the extra two conditions come in at $x = 1/2$. To derive the extra conditions, imagine integrating equation (36) across $x = 1/2$:

$$\int_{1/2-}^{1/2+} -y''(x) dx = \int_{1/2-}^{1/2+} \delta(x - 1/2) dx, \quad (39)$$

where $1/2-$ ($1/2+$) signifies just to the left (right) of $1/2$. Using property (31) of the delta function, we have

$$-y' \Big|_{1/2-}^{1/2+} = 1 \quad \Rightarrow \quad y'(1/2+) - y'(1/2-) = -1. \quad (40)$$

That is, the presence of the delta function defines a *jump condition* on y' .⁵ The other extra condition needed comes as a requirement that $y(x)$ is continuous across the point source, that is

$$y]_{1/2-}^{1/2+} = 0. \quad (41)$$

More on this condition below. Solving Equations (38), (37) along with extra conditions (40) and (41), we obtain the solution

$$u(x) = \begin{cases} \frac{x}{2} & 0 < x < 1/2 \\ -\frac{x}{2} + \frac{1}{2} & 1/2 < x < 1. \end{cases} \quad (42)$$

3.3.5 Green's function construction

To motivate the construction of the Green's function, consider the heat conduction problem with an arbitrary heat source:

$$-y''(x) = f(x), \quad 0 < x < 1 \quad (43)$$

$$y(0) = y(1) = 0. \quad (44)$$

Imagine now describing f by a distribution of point heat sources with varying strength; that is at point $x = \xi$ we imagine placing the point source $f(\xi)\delta(x - \xi)$.

The idea of the Green's function is to introduce such an extra parameter ξ , and consider the system

$$-g''(x, \xi) = \delta(x - \xi), \quad 0 < x < 1 \quad (45)$$

$$g(0, \xi) = g(1, \xi) = 0. \quad (46)$$

Note that prime denotes differentiation with respect to x , while ξ is more like a place-holding variable. So, we have replaced $f(x)$ by a delta function, in order to solve for the Green's function $g(x, \xi)$.

We have seen how to solve (45), (46) in the last section. The Green's function is

$$g(x, \xi) = \begin{cases} (1 - \xi)x & 0 < x < \xi \\ (1 - x)\xi & \xi < x < 1. \end{cases} \quad (47)$$

You can notice that this is exactly the solution (45) one would obtain via variation of parameters.

⁵Here, $y(\xi-) = \lim_{x \uparrow \xi} y(x)$, and $y(\xi+) = \lim_{x \downarrow \xi} y(x)$

How to get back to the solution of (43), (44)? For each ξ , the Green's function gives the solution if a point heat source of unit strength were placed at $x = \xi$. Conceptually, then, to get the full solution we must “add up” the point sources, scaled by the value of the heat source at each point:

$$y(x) = \int_0^1 g(x, \xi) f(\xi) d\xi. \quad (48)$$

To verify that this is indeed a solution, we can plug (48) into (43):

$$-y''(x) = \int_0^1 -g''(x, \xi) f(\xi) dx = \int_0^1 \delta(x - \xi) f(\xi) dx = f(x) \quad \checkmark \quad (49)$$

3.4 General linear BVP

We now consider a general n th order linear BVP with arbitrary continuous forcing function,

$$Ly(x) = a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_1 y'(x) + a_0 y(x) = f(x) \quad (50)$$

for $a < x < b$, where each $a_i = a_i(x)$ is a continuous function, and moreover $a_n(x) \neq 0 \forall x$ ⁶. Along with (50) are n boundary conditions, each a linear combination of y and derivatives up to $y^{(n-1)}$, evaluated at $x = a, b$. For instance, in the case $n = 2$, the general form is:

$$\begin{aligned} B_1 y &\equiv \alpha_{11} y(a) + \alpha_{12} y'(a) + \beta_{11} y(b) + \beta_{12} y'(b) = \gamma_1 \\ B_2 y &\equiv \alpha_{21} y(a) + \alpha_{22} y'(a) + \beta_{21} y(b) + \beta_{22} y'(b) = \gamma_2. \end{aligned} \quad (51)$$

3.5 General Green's Function

In the same way as in Section 3.3.4, to solve (50) with homogeneous BC

$$B_i y = 0, \quad i = 1 \dots n-1,$$

we first determine the Green's function by solving

$$\begin{aligned} Lg(x, \xi) &= \delta(x - \xi), \quad a < x < b \\ B_i g &= 0. \end{aligned} \quad (52)$$

⁶We'll return to the case where $a_n(x) = 0$ somewhere in the domain later in the course.

As before,

$$Lg(x, \xi) = \delta(x - \xi)$$

implies

$$Lg(x, \xi) = 0 \quad \text{on } a < x < \xi, \quad \xi < x < b,$$

i.e. we have a homogeneous problem to solve on two separate domains. As before, we require extra conditions, which come by integrating $Lg(x, \xi) = \delta(x - \xi)$ across $x = \xi$:

$$\int_{\xi-}^{\xi+} a_n g^{(n)}(x, \xi) + \cdots + a_0 g(x, \xi) \, d\xi = \int_{\xi-}^{\xi+} \delta(x - \xi) \, dx. \quad (53)$$

The right hand side clearly integrates to one. If we were to perform an integration by parts on the first term of the left hand side, we would obtain

$$a_n(x)g^{(n-1)}(x, \xi)]_{\xi-}^{\xi+} + \int_{\xi-}^{\xi+} (a_{n-1} - a'_n)g^{(n-1)} + \cdots + a_0 g(x, \xi) \, dx = 1.$$

This equation is balanced by setting a jump condition on the $n - 1$ st derivative:

$$g^{(n-1)}(x, \xi)]_{\xi-}^{\xi+} = 1/a_n(\xi),$$

and taking all lower derivatives to be continuous across $x = \xi$:

$$g^{(j)}(x, \xi)]_{\xi-}^{\xi+} = 0, \quad j = 0, 1, \dots, n - 2.$$

Once the Green's function is determined, the solution to the BVP is given by

$$y(x) = \int_a^b g(x, \xi) f(\xi) \, d\xi. \quad (54)$$

3.6 Another view

There is one more way of viewing the GF. Start from $Ly(x) = f(x)$, and take an inner product with $G(x, \xi)$ on both sides of the equation⁷. We are not assuming we know G , rather we want to find properties it should satisfy for us to solve the equation. We obtain

$$\langle Ly, G \rangle = \langle G(x, \xi), f(x) \rangle = \int_a^b G(x, \xi) f(x) \, dx. \quad (55)$$

⁷ G will be the Green's function, but not quite the same one we've constructed, so I am differentiating by using capital G .

(Note the integration is over x). Now, using the adjoint, we can write

$$\langle Ly, G \rangle = \langle y, L^*G \rangle \quad (56)$$

The idea now is to isolate y . This can be accomplished if

$$L^*G(x, \xi) = \delta(x - \xi) \quad (57)$$

in which case the left hand side leaves just $y(\xi)$, and we have the solution

$$y(\xi) = \int_a^b G(x, \xi) f(x) dx. \quad (58)$$

Comparing with our previous construction, here the big difference is that the GF is constructed through the adjoint operator – hence we will refer to this as the adjoint Green’s function. Compare the form of solution with the form (26):

$$y(x) = \int_a^b g(x, t) f(t) dt, \quad (59)$$

we see the subtle difference that in (58) we integrate over the first variable of the adjoint GF, and the second variable of the GF. For a self-adjoint operator, the constructions are the same and we must get the same GF, and indeed as we’ve stated, the GF for a self-adjoint operator is symmetric.

4 Distributions

We’ve seen in Section 3 how the Green’s function can be a valuable tool in solving BVPs. However, constructing the GF required defining the “delta function”, which is not really a function at all, at best a limit of functions, and also saw that the GF suffers a discontinuity in the $n - 1$ st derivative. We now take a short detour to consider these issues in more detail, by introducing the theory of distributions.

Perhaps the most important feature of the δ -“function”: when integrated against a continuous function, it sifts out the value at $x = 0$:

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0).$$

It is the operation of δ on another function that defines the property. This is the key idea in the theory of distributions, in which a generalized function

is only thought of in relation to how it affects other functions when “integrated” against them.

We define the delta distribution δ such that when it operates on a test function ϕ , it “sifts out” the value $\phi(0) \in \mathbb{R}$.

We write this as

$$\langle \delta, \phi \rangle \equiv \phi(0),$$

where δ is the δ -distribution and ϕ is the test function. $\langle \delta, \phi \rangle$ reads as “ δ applied to ϕ ”.

We will generalise this idea momentarily. First, we need some tools and terminology.

Test functions $\phi : \mathbb{R} \rightarrow \mathbb{R}$

$\phi \in C_0^\infty(\mathbb{R})$, which is short for:

- $\phi \in C^\infty(\mathbb{R})$ differentiable any number of times
- ϕ has “compact support”, i.e. $\text{supp } \phi \subseteq [-X, X]$ for some $X > 0$, i.e. $\phi(x) = 0 \quad \forall x \notin [-X, X]$.

So a test function is infinitely smooth, has no kinks or corners, and vanishes outside a finite region.

Example (see figure):

Let $C > 0$, $\epsilon > 0$

$$\phi_{C;\epsilon} = \begin{cases} \exp\left(\frac{-C}{\epsilon^2 - (x-a)^2}\right) & \text{for } a - \epsilon < x < a + \epsilon \\ 0 & \text{otherwise} \end{cases} \quad (60)$$

One can show (for all integer $n \geq 0$):

$$\lim_{x \uparrow a + \epsilon} \frac{d}{dx^n} \phi_{C;\epsilon}(x) = 0$$

$$\lim_{x \downarrow a - \epsilon} \frac{d}{dx^n} \phi_{C;\epsilon}(x) = 0$$

4.1 Weak derivative

Having defined test functions, we can generalise the notion of a derivative. Start with the classical definition: let $u(x)$ be a continuously differentiable

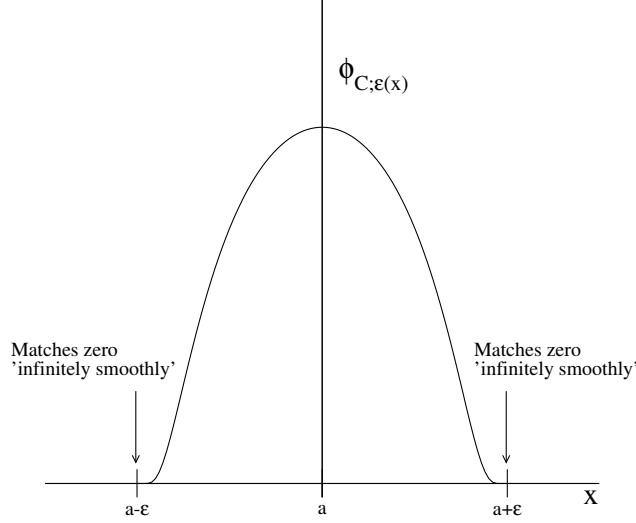


Figure 3: Sample test function, corresponding to (60)

function with derivative $f(x)$, so $u'(x) = f(x)$. Now, multiply each side of the equation by a test function ϕ and integrate over \mathbb{R} :

$$\int_{\mathbb{R}} u' \phi \, dx = \int_{\mathbb{R}} f \phi \, dx. \quad (61)$$

Integrating the LHS by parts and using the compact support of ϕ , we obtain

$$-\int_{\mathbb{R}} u \phi' \, dx = \int_{\mathbb{R}} f \phi \, dx. \quad (62)$$

The idea of the weak derivative is to think of (62) as the *definition* of a derivative. That is, we say f is the *weak derivative* of u if (62) holds for all test functions $\phi \in C_0^\infty(\mathbb{R})$ ⁸. The value is that this definition does not require u to be differentiable, just integrable. Of course, if u is continuously differentiable, the weak derivative and the ordinary one will agree, but a function that is not continuously differentiable can still have a weak derivative, where essentially the integration smooths out discontinuities.

⁸We can also confine to smaller intervals, for instance $\phi \in C_0^\infty(a, b)$ means the test functions have compact support in a bounded subset of (a, b) .

4.2 Distribution definition

This leads us to the notion of a distribution, or a generalised function. A distribution is not defined at points, but rather it is a global object defined in terms of its action on test functions. To be more precise:

Definition: A distribution u is a functional mapping test functions $\phi \in C_0^\infty(\mathbb{R})$ to real numbers,

$$u : \phi \in C_0^\infty(\mathbb{R}) \mapsto \langle u, \phi \rangle \in \mathbb{R} \quad (\langle u, \phi \rangle \text{ instead of } u(\phi)) \quad (63)$$

where the mapping is linear and continuous. While we have motivated the action $\langle u, \phi \rangle$ as meaning integration, this is not a requirement.

Linearity is straightforward, and means

$$\langle u, \alpha\phi + \beta\psi \rangle = \alpha\langle u, \phi \rangle + \beta\langle u, \psi \rangle \quad \forall \alpha, \beta \in \mathbb{R} \quad \forall \phi, \psi \in C_0^\infty(\mathbb{R}) \quad (64)$$

Continuity is slightly more technical, it means that if ϕ_n is a sequence of test functions that converges to zero,

$$\phi_n(x) \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$

then

$$\langle u, \phi_n \rangle \rightarrow 0 \quad (65)$$

as a sequence of real numbers.

To show continuity, what we really need is to be able to switch the order of “the action of the distribution” (integration) and the limit, that is (65) will hold if

$$\lim_{n \rightarrow \infty} \langle u, \phi_n \rangle = \langle u, \lim_{n \rightarrow \infty} \phi_n \rangle.$$

It turns out that we can do this if the following holds:

(*) $\forall X > 0$ there exists $C > 0$, and integer $N \geq 0$, such that

$$|\langle u, \phi \rangle| \leq C \sum_{m \leq N} \max_{-\infty \leq x \leq \infty} \left| \frac{d^m \phi}{dx^m} \right|$$

$\forall \phi$ with support in $[-X, X]$.

For our purposes we will want to show (*) to show continuity, and in fact you can take this as the definition of continuity.

Examples

Delta distribution

$$\langle \delta, \phi \rangle = \phi(0)$$

linearity: ✓

continuity, check (*): $|\langle \delta, \phi \rangle| = |\phi(0)| \leq \max_{-X < x < X} |\phi(x)| \quad \forall \phi$ with support of ϕ in $[-X, X]$.

i.e. condition (*) is satisfied with $C = 1$, $N = 0$.

Generalisation

Let $a \in \mathbb{R}$, $n \geq 0$. Define $\langle D_n, \phi \rangle = \phi^{(n)}(a)$ (n th derivative).

This is a distribution (to be proved in a problem sheet).

Functions as distributions. For any locally integrable function $f(x)$, a natural distribution is defined by

$$\langle f, \phi \rangle = \int_{-\infty}^{\infty} f(x)\phi(x)dx$$

Check:

Well-defined, $\langle f, \phi \rangle \in \mathbb{R} \quad \forall \phi \in C_0^\infty(\mathbb{R})$ and linear.

Continuity? (*): Let $X > 0$ be given. Claim (*) holds for

$$C = C(X) = \int_{-X}^X |f(x)|dx \text{ and } N = 0 :$$

$$|\langle f, \phi \rangle| = \left| \int_{-\infty}^{\infty} f(x)\phi(x)dx \right| = \left| \int_{-X}^X f(x)\phi(x)dx \right|$$

which by the estimation lemma

$$\leq \int_{-X}^X |f(x)|dx \max_{-X < x < X} (|\phi(x)|) = C \max_{-\infty < x < \infty} (|\phi(x)|)$$

Remark: Different *continuous* functions induce different distributions.

Heaviside function $H(x)$

$$\langle H, \phi \rangle = \int_{-\infty}^{\infty} H(x)\phi(x)dx = \int_0^{\infty} \phi(x)dx$$

Can check linearity, continuity as an exercise.

Remark: Different functions can lead to the same distribution.

Distributions induced by *integrable* functions are called *regular* distributions; singular distributions if not. The δ -distribution is an example of a singular distribution.

4.3 Operations on distributions

Now we consider some operations that can be performed on distributions. Let u_1, u_2, u be *distributions*, and f_1, f_2, f be integrable *functions* (or the regular distributions induced by them). The notion of integration is not required for distributions, but the rules for distributions are consistent with those for locally integrable functions.

Linear combinations of distributions. Let $\alpha_1, \alpha_2 \in \mathbb{R}$.

$$\begin{aligned} \langle \alpha_1 f_1 + \alpha_2 f_2, \phi \rangle &= \int_{-\infty}^{\infty} (\alpha_1 f_1(x) + \alpha_2 f_2(x))\phi(x)dx \\ &= \alpha_1 \int_{-\infty}^{\infty} f_1(x)\phi(x)dx + \alpha_2 \int_{-\infty}^{\infty} f_2(x)\phi(x)dx \\ &= \alpha_1 \langle f_1, \phi \rangle + \alpha_2 \langle f_2, \phi \rangle \end{aligned}$$

Thus, define $\alpha_1 u_1 + \alpha_2 u_2$ for general distributions u_1, u_2 via

$$\langle \alpha_1 u_1 + \alpha_2 u_2, \phi \rangle \equiv \alpha_1 \langle u_1, \phi \rangle + \alpha_2 \langle u_2, \phi \rangle \quad \forall \phi \in C_0^\infty(\mathbb{R})$$

If u_1, u_2 are distributions, is $\alpha_1 u_1 + \alpha_2 u_2$ a distribution? Need to check linearity and continuity, but we'll skip this here.

Differentiation of distributions. Differentiation follows the weak derivative formulated earlier. That is, for a general distribution u , define

$$\langle u', \phi \rangle \equiv -\langle u, \phi' \rangle \quad \forall \phi \in C_0^\infty(\mathbb{R})$$

If u is distribution, can we be sure that $u' : \phi \mapsto -\langle u, \phi' \rangle$ is also a distribution? (It is! – try it as an exercise.)

Example. Let H be the Heaviside function, or the distribution it induces, i.e.

$$\langle \underbrace{H}_{H\text{-distribution}}, \phi \rangle \equiv \int_{-\infty}^{\infty} \underbrace{H(x)}_{H\text{-function}} \phi(x) dx = \int_0^{\infty} \phi(x) dx$$

Show that $H' = \delta$.

$$\begin{aligned} \langle H', \phi \rangle &= \langle -H, \phi' \rangle && \text{Def. of derivative of a distribution} \\ &= \int_{-\infty}^{\infty} \phi'(x) dx && \text{see earlier example} \\ &= -\phi|_{x=0}^{x=\infty} \\ &= \phi(0) && \phi \text{ has compact support} \\ &= \langle \delta, \phi \rangle && \text{Def. of } \delta\text{-distribution} \end{aligned}$$

Translation: similar considerations as before, upshot ($a \in \mathbb{R}, u$ distr):

$$\langle u(x-a), \phi(x) \rangle \stackrel{\text{chg of var}}{=} \langle u(y), \phi(y+a) \rangle = \langle u(x), \phi(x+a) \rangle$$

Example: $\langle \delta(x-a), \phi(x) \rangle = \langle \delta(x), \phi(x+a) \rangle = \phi(a)$

Multiplication: let $a(x)$ be an infinitely differentiable function. We define

$$\langle au, \phi \rangle = \langle u, a\phi \rangle.$$

Convergence of a sequence of distributions u, u_1, u_2, \dots distributions.

Convergence $u_j \rightarrow u$ as $j \rightarrow \infty$ means:

$$\lim_{j \rightarrow \infty} \langle u_j, \phi \rangle = \langle u, \phi \rangle \quad \forall \phi \in C_0^\infty(\mathbb{R})$$

Similarly: if $u(\alpha)$ is a family of distributions with a continuous parameter α , then

convergence $u(\alpha) \rightarrow u(\alpha_0)$ for $\alpha \rightarrow \alpha_0$ means:

$$\lim_{\alpha \rightarrow \alpha_0} \langle u(\alpha), \phi \rangle = \langle u(\alpha_0), \phi \rangle \quad \forall \phi \in C_0^\infty(\mathbb{R})$$

4.4 Distributed solutions

Consider the equation

$$Lu \equiv a_2 u'' + a_1 u' + a_0 u = f.$$

We have always thought about the *classical solution*, that is a twice continuously differentiable function $u(x)$ that satisfies the differential equation identically, i.e. we can take derivatives of u , substitute in, and the equation checks at every point. With distribution theory and the notion of a generalised function, we now can define a *distributed solution*. That is, if u and f are distributions, then Lu is a distribution, defined by the action

$$\begin{aligned}\langle Lu, \phi \rangle &= \langle a_2 u'', \phi \rangle + \langle a_1 u', \phi \rangle + \langle a_0 u, \phi \rangle \\ &= \langle u, (a_2 \phi)'' \rangle - \langle u, (a_1 \phi)' \rangle + \langle u, a_0 \phi \rangle \stackrel{\text{define}}{=} \langle u, L^* \phi \rangle.\end{aligned}\tag{66}$$

Here L^* is the formal adjoint operator. We say that u is a distributed solution to $Lu = f$ if

$$\langle u, L^* \phi \rangle = \langle f, \phi \rangle$$

holds for all test functions ϕ . We highlight again that a function need not be differentiable in the ordinary sense to satisfy this definition; hence, distributions provide a way to have well-defined solutions that may have issues in the classical sense.

In particular, this construction of a distributed solution gives us a new way to interpret the Green's function. Since δ is really a distribution or a generalised function, the equation $Lg = \delta(x - \xi)$ should be interpreted in the distributional sense,

$$\langle Lg, \phi \rangle = \langle \delta(x - \xi), \phi \rangle$$

or

$$\langle g(x, \xi), L^* \phi \rangle = \phi(\xi).$$

Moreover, since the Green's function that we construct is not twice continuously differentiable, it is really a distributed solution. Alternatively, if we interpret $Lg = \delta(x - \xi)$ as meaning that $Lg = 0$ everywhere that $x \neq \xi$, then using the properties of δ we can work purely in the “classical” sense. In fact, the final solution of $Ly = f$, obtained by integration with g , is continuous and a classical solution.

Final thoughts: If you are interested in distribution theory, it is at the core of functional analysis. Moreover, the idea of weak formulations has great use in finite element methods. For us, distribution theory is somewhat of a detour for this course. One could proceed to write things in a distributional sense anytime we encounter a ‘delta function’, but we can as well recognise delta as the limit of continuous functions and satisfying certain properties,

thus in effect translating to a classical system. Unless we are specifically interested in a distributional aspect, the latter will be our approach.