

Problem Sheet 4

Distributions

1. Scalar property of delta function. Show from its interpretation as an integral that $\delta(ax) = \frac{1}{|a|}\delta(x)$ for any constant a .
2. Convergence of distributions. Show that the following function sequences converge to the δ -distribution as $n \rightarrow \infty$.

(a)
$$f_n(x) = \begin{cases} n/2 & \text{for } -1/n < x < 1/n, \\ 0 & \text{else} \end{cases}$$

(b)
$$f_n(x) = \frac{e^{-nx^2/4}}{\sqrt{4\pi/n}} \quad (\text{using } \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \text{ without proof})$$

You have to show that $\lim_{n \rightarrow \infty} \langle f_n, \phi \rangle = \langle \delta, \phi \rangle$ holds for all $\phi \in C_0^\infty(\mathbb{R})$, i.e. that $\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f_n(x)\phi(x)dx = \phi(0)$

Hints: In (a), split the integral as

$$\int_{-\infty}^{\infty} f_n(x)\phi(x)dx = \int_{-\infty}^{\infty} f_n(x) [\phi(x) - \phi(0)] dx + \phi(0) \int_{-\infty}^{\infty} f_n(x)dx.$$

The second integral can be evaluated explicitly. What can you say about the first? A similar approach for (b) though a bit more work. In my model solution I split the integral into 4 parts, including a term

$$\int_{-\rho}^{\rho} f_n(x) [\phi(x) - \phi(0)] dx$$

with ρ chosen appropriately.

3. Derivatives of distributions.

(a) For $n \geq 0$, $a \in \mathbb{R}$, let $D_n : C_0^\infty \rightarrow \mathbb{R}$ be defined by

$$\langle D_n, \phi \rangle = \left. \frac{d^n \phi}{dx^n} \right|_{x=a} = \phi^{(n)}(a) \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}).$$

Show that D_n is a distribution, and then, by induction, that

$$D_n = (-1)^n \delta^{(n)}(x - a),$$

where $\delta^{(n)}$ is the n -th distributional derivative of the δ -distribution.

- (b) Let $T : C_0^\infty \rightarrow \mathbb{R}$ be a distribution (not necessarily one that is induced by a continuous function). Using the definition of distributional derivative, translation of distributions and convergence of distributions, show that

$$\lim_{\alpha \rightarrow 0} \frac{T(x + \alpha) - T(x)}{\alpha} = T'(x)$$

You may use (without proof) that for any $\phi \in C_0^\infty(\mathbb{R})$,

$$\rho_\alpha(x) \equiv \frac{\phi(x) - \phi(x - \alpha)}{\alpha}$$

converges to $\phi'(x)$ uniformly in x as $\alpha \rightarrow 0$.