

MMSC Further Mathematical Methods HT2016 — Sheet 2

1. For each of the following problems (a)–(e), use the Fredholm alternative to state the conditions under which it has a unique solution, the conditions under which it has no solution, and the conditions under which it has multiple solutions. Here α , β and b are continuous, real-valued functions.

(a) $u''(x) - u(x) = b(x)$ on $0 < x < 1$, with $u(0) = u(1) = 0$.

(b) $u''(x) + u(x) = b(x)$ on $0 < x < L$, with $u(0) = u(L) = 0$, where $L > 0$ is a given constant (possibly $n\pi$).

(c) $u''(x) + \alpha(x)u'(x) + \beta(x)u(x) = b(x)$ on $0 < x < 1$, with $u(0) = u'(0) = 0$.

(d) $u'(x) + \alpha(x)u(x) = b(x)$ on $0 < x < 1$, with $u(0) = u(1)$.

2. Consider the problem

$$\epsilon \frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} - \sin f + \epsilon, \quad f' \rightarrow 0 \text{ as } |x| \rightarrow \infty,$$

with $f(-\infty)$ close to 0 and $f(\infty)$ close to 2π . Show that there exists a travelling wave solution

$$f \sim \phi(x - ct)$$

where

$$\frac{d^2\phi}{dx^2} = \sin \phi, \quad \phi \rightarrow 0 \text{ as } x \rightarrow -\infty, \quad \phi \rightarrow 2\pi \text{ as } x \rightarrow \infty, \quad \phi(0) = \pi,$$

and determine the wavespeed c to leading order in ϵ . [In fact $\phi(x) = 4 \tan^{-1} e^x$.]

3. Consider the problem

$$\omega^2 \ddot{x} + \epsilon(\lambda - x^2)\dot{x} + x = 0, \quad x(0) = x(2\pi), \quad \dot{x}(0) = \dot{x}(2\pi).$$

By expanding

$$x(t) \sim x_0(t) + \epsilon x_1(t) + \dots, \quad \omega \sim 1 + \epsilon \omega_1 + \dots,$$

Show that there is a solution

$$x_0 = A \cos t,$$

providing A satisfies a solvability condition, which you should determine.