MMSC Further Mathematical Methods HT2017 — Sheet 4

1. If $\dot{x}(t) = tu(t)$ and $x(0) = 0$ and $x(1) = 1$, show that the control u that minimises the cost

$$C[u] = \int_0^1 u(t)^2 \, \mathrm{d}t$$

is given by $u(t) = 3t$, and find the corresponding behaviour of $x(t)$.

2. Suppose the functions $x(t)$ and $u(t)$ satisfy the differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x, u), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial h/\partial u}{\partial f/\partial u}\right) = \frac{\partial h}{\partial x} - \left(\frac{\partial h/\partial u}{\partial f/\partial u}\right)\frac{\partial f}{\partial x},$$

where $f(t, x, u)$ and $h(t, x, u)$ are given smooth functions. Show that the Hamiltonian

$$H = \left(\frac{\partial h/\partial u}{\partial f/\partial u}\right) f - h$$

is conserved when the problem is *autonomous*, that is when $\partial f/\partial t = \partial h/\partial t = 0$.

3. It is required to minimise the cost function

$$C[x, u] = \int_0^1 \left(x(t)^2 + u(t)^2\right) \mathrm{d}t$$

over all control functions $u(t)$ which raise the solution of the differential equation $\dot{x} = x^2 + u$ from $x = 0$ at $t = 0$ to $x = 1$ at $t = 1$. Show that, for this choice of $u(t)$, the function $x(t)$ is given by the implicit equation

$$t = \int_0^x \left(\xi^4 + \xi^2 + H\right)^{-1/2} \mathrm{d}\xi,$$

where H is a constant, and state a condition that determines its value.

4. A process obeys the first-order differential equation $\dot{x} = u - x$ with $x(0) = a$, and it is desired to minimise the integral

$$C[x, u] = \int_0^T \left(x(t)^2 + u(t)^2\right) \mathrm{d}t.$$

Show that the optimal control takes the form $u(t) = -K(t)x(t)$, where the gain $K(t)$ satisfies the Riccati equation

$$\dot{K} = K^2 + 2K - 1.$$

Specify a boundary condition for K and hence calculate $K(t)$ explicitly.

5. A colony of insects consists of workers and queens, of numbers $w(t)$ and $q(t)$ at time t. If a time-dependent proportion $u(t)$ of the colony's effort is put into producing workers ($0 \leq u(t) \leq 1$), then w and q obey the differential equations

$$\dot{w} = auw - w, \quad \dot{q} = c(1-u)w,$$

where a and c are positive constants with $a > 1$. The function u is to be chosen so as to maximize the number of queens at the end of the season. Show that the optimal policy is to produce only workers up to some moment, and only queens thereafter. Show that the optimal switching time is $\log(a/(a-1))$ before the end of the season.

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