1 Eigenfunction Expansions

1.1 Matrices

Consider the linear system of equations

$$A\mathbf{x} = \mathbf{b}, \tag{1.1}$$

where A is a real symmetric $N \times N$ matrix and $\mathbf{x}$ and $\mathbf{b}$ are column vectors of length N.

The right eigenvectors $\mathbf{e}_n$ of A, with corresponding eigenvalues λ_n, satisfy

$$A\mathbf{e}_n = \lambda_n \mathbf{e}_n.$$

These are orthogonal and by an appropriate scaling may be made orthonormal, so that

$$\mathbf{e}_n^T \mathbf{e}_m = \delta_{nm}$$

where

$$\delta_{nm} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases}$$

is the Kronecker delta.

[Proof: consider $\mathbf{e}_m^T A \mathbf{e}_n = \mathbf{e}_m^T \lambda_n \mathbf{e}_n = (A^T \mathbf{e}_m)^T \mathbf{e}_n = (A\mathbf{e}_m)^T \mathbf{e}_n = \lambda_m \mathbf{e}_m^T \mathbf{e}_n$. If $\lambda_n \neq \lambda_m$ then $\mathbf{e}_m^T \mathbf{e}_n = 0$. For eigenvalues of multiplicity greater than one choose an orthonormal basis of the corresponding eigenspace.]

Proposition 1.1. We can write the identity matrix I as

$$I = \sum_{n=1}^{N} \mathbf{e}_n \mathbf{e}_n^T.$$

Proof.

$$\left(\sum_{n=1}^{N} \mathbf{e}_n \mathbf{e}_n^T \right) \mathbf{e}_m = \sum_{n=1}^{N} \mathbf{e}_n \delta_{nm} = \mathbf{e}_m = I\mathbf{e}_m.$$

Since the vectors $\mathbf{e}_m$ form a basis we conclude that

$$\sum_{n=1}^{N} \mathbf{e}_n \mathbf{e}_n^T = I.$$

□

Proposition 1.2. We can write A as

$$A = \sum_{n=1}^{N} \lambda_n \mathbf{e}_n \mathbf{e}_n^T.$$

Proof. Again, we consider the action of the matrix on the basis vectors $\mathbf{e}_m$.

$$\left(\sum_{n=1}^{N} \lambda_n \mathbf{e}_n \mathbf{e}_n^T \right) \mathbf{e}_m = \sum_{n=1}^{N} \lambda_n \mathbf{e}_n \delta_{nm} = \lambda_m \mathbf{e}_m = A\mathbf{e}_m.$$

□

Proposition 1.3. We can write A^{-1} as

$$A^{-1} = \sum_{n=1}^{N} \frac{\mathbf{e}_n \mathbf{e}_n^T}{\lambda_n}.$$

Proof. We left multiply equation (1.1) by $\mathbf{e}_n^T$:

$$\begin{aligned} & \mathbf{e}_n^T A\mathbf{x} = \mathbf{e}_n^T \mathbf{b} \\ \Rightarrow \quad & (A^T \mathbf{e}_n)^T \mathbf{x} = \mathbf{e}_n^T \mathbf{b} \\ \Rightarrow \quad & (A\mathbf{e}_n)^T \mathbf{x} = \mathbf{e}_n^T \mathbf{b} \\ \Rightarrow \quad & \lambda_n \mathbf{e}_n^T \mathbf{x} = \mathbf{e}_n^T \mathbf{b} \end{aligned}$$

If $\lambda_n \neq 0$ then

$$\mathbf{e}_n^T \mathbf{x} = \frac{1}{\lambda_n} \mathbf{e}_n^T \mathbf{b}$$

Now

$$\mathbf{x} = A^{-1}\mathbf{b} = \sum_{n=1}^{N} \mathbf{e}_n \mathbf{e}_n^T \mathbf{x} = \sum_{n=1}^{N} \mathbf{e}_n \frac{1}{\lambda_n} \mathbf{e}_n^T \mathbf{b} = \left(\sum_{n=1}^{N} \frac{\mathbf{e}_n \mathbf{e}_n^T}{\lambda_n} \right) \mathbf{b}$$

Since $\mathbf{b}$ is arbitrary the proposition is proved. □

This is just another way of writing

$$A = \left(\mathbf{e}_1 \,\middle|\, \cdots \,\middle|\, \mathbf{e}_N \right) \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_N \end{pmatrix} \begin{pmatrix} \mathbf{e}_1^T \\ - \\ \vdots \\ - \\ \mathbf{e}_N^T \end{pmatrix} = E\Lambda E^T,$$

$$\begin{aligned} A^{-1} &= (E\Lambda E^T)^{-1} = (E^T)^{-1}\Lambda^{-1}E^{-1} = E\Lambda^{-1}E^T \\ &= \left(\mathbf{e}_1 \,\middle|\, \cdots \,\middle|\, \mathbf{e}_N \right) \begin{pmatrix} \lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_N^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{e}_1^T \\ - \\ \vdots \\ - \\ \mathbf{e}_N^T \end{pmatrix}, \end{aligned}$$

Note

$$E^T E = EE^T = I.$$

Exercise (problem sheet 1) Suppose $\lambda_1 = 0$, $\lambda_n \neq 0$ for $n \neq 1$. Show that when $\mathbf{e}_1^T\mathbf{b} \neq 0$ there is no solution to $A\mathbf{x} = \mathbf{b}$, while when $\mathbf{e}_1^T\mathbf{b} = 0$ the solution is

$$\mathbf{x} = \sum_{n=2}^{N} \frac{\mathbf{e}_n\mathbf{e}_n^T}{\lambda_n}\mathbf{b} + \alpha\mathbf{e}_1,$$

where α is arbitrary.

1.2 Functions

Consider the Sturm-Liouville problem

$$Lu = f$$

for functions $u(x)$ with $x \in [a, b]$ with suitable boundary conditions, where L is self-adjoint, so that

$$\int_a^b v(x)Lu(x)\,\mathrm{d}x = \int_a^b u(x)Lv(x)\,\mathrm{d}x.$$

Let us use the notation

$$\langle u, v\rangle = \int_a^b u(x)v(x)\,\mathrm{d}x.$$

Sturm-Liouville problems have complete sets of orthonormal eigenfunctions $\phi_n(x)$ such that

$$L\phi_n = \lambda_n \phi_n$$

and

$$\langle \phi_n, \phi_m \rangle = \delta_{nm}.$$

[Why?

$$\langle \phi_m, L\phi_n \rangle = \lambda_n \langle \phi_m, \phi_n \rangle = \langle L\phi_m, \phi_n \rangle = \lambda_m \langle \phi_n, \phi_m \rangle.$$

Thus

$$(\lambda_n - \lambda_m)\langle \phi_n, \phi_m \rangle = 0,$$

so that $\lambda_n \neq \lambda_m$ implies $\langle \phi_n, \phi_m \rangle = 0$.]

How do we approximate a function f using the eigenfunctions ϕ_n? Let us minimise

$$\begin{aligned}
||f - \sum_i \alpha_i \phi_i||^2 &= \langle f - \sum_i \alpha_i \phi_i, f - \sum_j \alpha_j \phi_j \rangle \\
&= \langle f, f - \sum_j \alpha_j \phi_j \rangle - \sum_i \alpha_i \langle \phi_i, f - \sum_j \alpha_j \phi_j \rangle \\
&= \langle f, f \rangle - \sum_j \alpha_j \langle f, \phi_j \rangle - \sum_i \alpha_i \langle \phi_i, f \rangle + \sum_i \sum_j \alpha_i \alpha_j \langle \phi_i, \phi_j \rangle \\
&= ||f||^2 - 2\sum_i \alpha_i \langle f, \phi_i \rangle + \sum_i \alpha_i^2.
\end{aligned}$$

Now complete the square on α_i to give

$$||f - \sum_i \alpha_i \phi_i||^2 = \sum_i (\alpha_i - \langle f, \phi_i \rangle)^2 + ||f||^2 - \sum_i \langle f, \phi_i \rangle^2.$$

The only dependence on α_i is in the first term, which is minimised by taking $\alpha_i = \langle f, \phi_i \rangle$. This leaves

$$0 \leq ||f - \sum_i \alpha_i \phi_i||^2 = ||f||^2 - \sum_i \langle f, \phi_i \rangle^2.$$

This gives Bessel's inequality:

$$||f||^2 \geq \sum_i \langle f, \phi_i \rangle^2.$$

If the ϕ_i are complete then $||f - \sum_i \alpha_i \phi_i||^2 = 0$, giving Parseval's theorem:

$$||f||^2 = \sum_i \langle f, \phi_i \rangle^2.$$

Thus we have

$$\begin{aligned} f(x) &= \sum_i \alpha_i \phi_i(x) = \sum_i \langle f, \phi_i \rangle \phi_i(x) \\ &= \sum_i \int_a^b f(\xi)\phi_i(\xi)\,\mathrm{d}\xi \phi_i(x) = \int_a^b \left(\sum_i \phi_i(\xi)\phi_i(x) \right) f(\xi)\,\mathrm{d}\xi. \end{aligned}$$

But this means that

$$\sum_i \phi_i(\xi)\phi_i(x) = \delta(x-\xi).$$

This is the equivalent of

$$\sum_n \mathbf{e}_n \mathbf{e}_n^T = I$$

for the case of matrices.

Now we can solve the linear equation

$$Lu = f$$

(with the same boundary conditions as before). Taking the inner product with ϕ_n gives

$$\langle \phi_n, f \rangle = \langle \phi_n, Lu \rangle = \langle L\phi_n, u \rangle = \lambda_n \langle \phi_n, u \rangle.$$

Therefore

$$\langle \phi_n, u \rangle = \frac{\langle \phi_n, f \rangle}{\lambda_n}$$

provided $\lambda_n \neq 0$. Then

$$u = \sum_n \langle \phi_n, u \rangle \phi_n = \sum_i \frac{1}{\lambda_n} \langle \phi_n, f \rangle \phi_n.$$

Thus

$$\begin{aligned} u(x) &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \int_a^b \phi_n(\xi) f(\xi)\,\mathrm{d}\xi \phi_n(x) \\ &= \int_a^b \left(\sum_{n=1}^{\infty} \frac{\phi_n(\xi)\phi_n(x)}{\lambda_n} \right) f(\xi)\,\mathrm{d}\xi \\ &= \int_a^b G(x,\xi) f(\xi)\,\mathrm{d}\xi \end{aligned}$$

where

$$G(x,\xi) = \sum_{n=1}^{\infty} \frac{\phi_n(\xi)\phi_n(x)}{\lambda_n} = G(\xi, x).$$

Thus is the equivalent of

$$A^{-1} = \sum_{n=1}^{N} \frac{\mathbf{e}_n \mathbf{e}_n^T}{\lambda_n} = (A^{-1})^T$$

for real symmetric matrices.

We should not be surprised by this expression for G. For any function $u = \sum_{n=1}^{\infty} \alpha_n \phi_n$ with $\alpha_n = \langle u, \phi_n \rangle$,

$$Lu = \sum_{n=1}^{\infty} \alpha_n \lambda_n \phi_n,$$

$$L^2 u = L\left(\sum_{n=1}^{\infty} \alpha_n \lambda_n \phi_n\right) = \sum_{n=1}^{\infty} \alpha_n \lambda_n^2 \phi_n,$$

and so

$$L^{(k)} u = \sum_{n=1}^{\infty} \alpha_n \lambda_n^k \phi_n,$$

for any k. Thus, if $g(x)$ is a polynomial we can define the operator $g(L)$ by

$$g(L)u = \sum_{n=1}^{\infty} \alpha_n g(\lambda_n) \phi_n.$$

If g is analytic the same definition is reasonable. Thus the inverse operator $g(L) = L^{-1}$ is

$$L^{-1} u = \sum_{n=1}^{\infty} \alpha_n \lambda_n^{-1} \phi_n.$$

Thus

$$L^{-1} f = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle \lambda_n^{-1} \phi_n = \sum_{n=1}^{\infty} \left(\int_{-a}^{b} f(\xi) \phi_n(\xi) \, \mathrm{d}\xi \right) \frac{\phi_n(x)}{\lambda_n} = \int_{-a}^{b} \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(\xi)}{\lambda_n} f(\xi) \, \mathrm{d}\xi.$$

Example 1.1. Waves on a string Consider

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = f(x) \cos(\omega t)$$

for $0 \le x \le \pi, \quad t \ge 0$, with $u(0,t) = u(\pi,t) = 0$. Try $u(x,t) = U(x)\cos(\omega t)$ with $U(0) = U(\pi) = 0$. This gives the forced Helmholtz equation

$$\frac{\mathrm{d}^2 U}{\mathrm{d}x^2} + k^2 U = f$$

with $k = \omega/c \notin \mathbb{Z}$. Thus

$$LU = \frac{\mathrm{d}^2U}{\mathrm{d}x^2} + k^2U \qquad \text{with } U(0) = U(\pi) = 0.$$

The eigenfunctions are

$$\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx),$$

with eigenvalues

$$\lambda_n = k^2 - n^2, \qquad \text{for } n = 1, 2, \ldots$$

Thus the Green's function is

$$G(x,\xi) = \sum_{n=1}^{\infty}\frac{\phi_n(x)\phi_n(\xi)}{\lambda_n} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)\sin(n\xi)}{k^2-n^2},$$

and

$$U(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{k^2-n^2}\int_0^{\pi}\sin(n\xi)f(\xi)\,\mathrm{d}\xi\right)\sin(nx).$$

If we solve

$$LG = \delta(x-\xi), \qquad G(0) = G(\pi) = 0$$

we find

$$G(x,\xi) = \begin{cases} -\dfrac{\sin(kx)\sin(k(\pi-\xi))}{k\sin k\pi} & 0 \le x < \xi < \pi \\ -\dfrac{\sin(k\xi)\sin(k(\pi-x))}{k\sin k\pi} & 0 \le \xi < x \le \pi \end{cases}$$

(see problem sheet 1). How can we relate these two expressions for G? Consider

$$F(k) = -\frac{\sin(kx)\sin(k(\pi-\xi))}{k\sin k\pi}$$

for $x < \xi$ as a function of k. A general complex function $F(k)$ with poles but no branch points can always be written

$$F(k) = \sum_n \frac{\operatorname{res}(F,k_n)}{k-k_n} + \text{ entire function} \tag{1.2}$$

where $\operatorname{res}(F,k_n)$ is the residue of F at the pole k_n. In this case our $F(k)$ has simple poles at $k = \pm1, \pm2, \ldots$ and a removable singularity at $k = 0$. To find the residue at

$k = n$ put $k = n + \epsilon$ and expand for ϵ small

$$\begin{aligned} F(n+\epsilon) &= -\frac{\sin((n+\epsilon)x)\sin((n+\epsilon)(\pi-\xi))}{(n+\epsilon)\sin(n+\epsilon)\pi} \\ &\sim -\frac{\sin(nx)\sin(n(\pi-\xi))}{n(-1)^n\epsilon\pi} \\ &\sim \frac{\sin(nx)\sin(n\xi)}{n\pi\epsilon}. \end{aligned}$$

Thus

$$\mathrm{res}(F,n) = \frac{\sin(nx)\sin(n\xi)}{n\pi}.$$

To eliminate the entire function in (1.2) we need to estimate the behaviour of G as $|k| \to \infty$. We find

$$|G| \sim \frac{1}{2|k|}\frac{\mathrm{e}^{|\mathrm{Im}(k)|x}\mathrm{e}^{|\mathrm{Im}(k)|(\pi-\xi)}}{\mathrm{e}^{|\mathrm{Im}(k)|\pi}} = \frac{\mathrm{e}^{|\mathrm{Im}(k)|(x-\xi)}}{2|k|} \to 0$$

as $|k| \to \infty$ since $x < \xi$. Thus the entire function is zero by Liouville's theorem. Therefore, for $x < \xi$,

$$\begin{aligned} G(x,\xi) = \sum_{n\neq 0}\frac{\sin(nx)\sin(n\xi)}{n\pi(k-n)} &= \sum_{n=1}^{\infty}\left(\frac{\sin(nx)\sin(n\xi)}{n\pi(k-n)} - \frac{\sin(nx)\sin(n\xi)}{n\pi(k+n)}\right) \\ &= \sum_{n=1}^{\infty}\frac{\sin(nx)\sin(n\xi)}{n\pi}\left(\frac{1}{(k-n)} - \frac{1}{(k+n)}\right) \\ &= \sum_{n=1}^{\infty}\frac{\sin(nx)\sin(n\xi)}{n\pi}\frac{2n}{k^2-n^2} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nx)\sin(n\xi)}{k^2-n^2}, \end{aligned}$$

as required.

■

2 Fourier Transform

2.1 Continuous spectrum

Consider the modified Helmholtz equation

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \alpha^2 U = f(x) \tag{2.1}$$

with U bounded as $|x| \to \infty$. The corresponding eigenvalue problem is

$$LU = \frac{\mathrm{d}^2U}{\mathrm{d}x^2} - \alpha^2 U = \lambda U.$$

The bounded solutions are

$$U = \mathrm{e}^{\mathrm{i}kx}$$

with corresponding eigenvalue $\lambda(k) = -(k^2 + \alpha^2)$. In this case k is not limited to a discrete set but can take any real value.

Let us solve equation (2.1) by Fourier transform. Define

$$\tilde{U}(k) = \int_{-\infty}^{\infty} U(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x.$$

Then, applying this operator to equation (2.1),

$$-k^2\tilde{U} - \alpha^2\tilde{U} = \tilde{f}.$$

Thus

$$\tilde{U} = -\frac{\tilde{f}}{k^2 + \alpha^2}. \tag{2.2}$$

Inverting the transform gives

$$\begin{aligned} U(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{U}(k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\tilde{f}(k)}{-(k^2+\alpha^2)}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(\xi)}{-(k^2+\alpha^2)}\mathrm{e}^{\mathrm{i}k\xi}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}\xi\,\mathrm{d}k \\ &= \int_{-\infty}^{\infty} G(x,\xi)f(\xi)\,\mathrm{d}\xi \end{aligned}$$

where

$$G(x,\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}k\xi}\mathrm{e}^{-\mathrm{i}kx}}{-(k^2+\alpha^2)}\,\mathrm{d}k = \int_{-\infty}^{\infty} \frac{\phi_k^*(\xi)\phi_k(x)}{\lambda(k)}\,\mathrm{d}k$$

where

$$\phi_k(x) = \frac{\mathrm{e}^{-\mathrm{i}kx}}{\sqrt{2\pi}} \qquad \text{and} \qquad \lambda(k) = -(k^2+\alpha^2).$$

This is the continuous version of

$$G(x,\xi) = \sum_{n=1}^{\infty} \frac{\phi_n(\xi)\phi_n(x)}{\lambda_n},$$

in which the sum over n becomes an integral over k.

The Fourier transform is an expansion in the eigenfunctions

$$\phi_k(x) = \frac{\mathrm{e}^{-\mathrm{i}kx}}{\sqrt{2\pi}}$$

of $\mathrm{d}/\mathrm{d}x$ with eigenvalues $-\mathrm{i}k$.

Note that it is possible for an eigenvalue problem to have bother a continuous and discrete spectrum. For example, the Schrödinger equation

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\right)\Psi = \lambda\Psi$$

with $V(x) \le 0$ and $V(x) \to 0$ as $|x| \to \infty$ has a discrete spectrum with $\Psi_n \sim \mathrm{e}^{-k_n|x|}$ as $|x| \to \infty$ and a continuous spectrum with $\Psi_k \sim \mathrm{e}^{\mathrm{i}kx}$ as $|x| \to \infty$.

2.2 Fourier Transform

We saw above that the Fourier transform is an expansion in the eigenfunctions

$$\phi_k(x) = \frac{\mathrm{e}^{-\mathrm{i}kx}}{\sqrt{2\pi}}$$

of $\mathrm{d}/\mathrm{d}x$ with eigenvalues $-\mathrm{i}k$. The pure imaginary eigenvalue ensures the eigenfunctions are bounded as $x \to \pm\infty$.

The forward Fourier transform is

$$F(k) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x.$$

The inverse Fourier transform is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k.$$

Putting these two together gives

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\,\mathrm{e}^{-\mathrm{i}kx}\int_{-\infty}^{\infty} \mathrm{d}\xi\,\mathrm{e}^{\mathrm{i}k\xi}f(\xi) = \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}kx}\mathrm{e}^{\mathrm{i}k\xi}\,\mathrm{d}k\right) f(\xi)\mathrm{d}\xi.$$

Thus

$$\delta(x-\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}k(x-\xi)}\,\mathrm{d}k = \int_{-\infty}^{\infty}\frac{\mathrm{e}^{-\mathrm{i}kx}}{\sqrt{2\pi}}\frac{\mathrm{e}^{\mathrm{i}k\xi}}{\sqrt{2\pi}}\,\mathrm{d}k = \int_{-\infty}^{\infty}\phi_k(x)\phi_k^*(\xi)\,\mathrm{d}k.$$

Note that the inverse Fourier transform is really an eigenfunction expansion in the eigenfunctions of $\mathrm{d}/\mathrm{d}x$:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k = \int_{-\infty}^{\infty}\frac{F(k)}{\sqrt{2\pi}}\phi_k(x)\,\mathrm{d}k$$

where

$$\phi_k(x) = \frac{\mathrm{e}^{-\mathrm{i}kx}}{\sqrt{2\pi}}.$$

Compare to

$$f(x) = \sum_n \langle f, \phi_n\rangle\phi_n(x)$$

in the discrete case. In fact, to complete the correspondence, we should use the symmetric Fourier transform, in which the forward Fourier transform is

$$\hat{F}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\phi_k^*(x)\,\mathrm{d}x$$

and the inverse Fourier transform is

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{F}(k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k = \int_{-\infty}^{\infty}\hat{F}(k)\phi_k(x)\,\mathrm{d}k.$$

2.2.1 Properties of Fourier Transforms

If $g(x) = f'(x)$ then

$$G(k) = \int_{-\infty}^{\infty} f'(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = \left[f(x)\mathrm{e}^{\mathrm{i}kx}\right]_{-\infty}^{\infty} - \mathrm{i}k\int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = -\mathrm{i}kF(k).$$

Thus differentiation becomes multiplication by $-\mathrm{i}k$ (the eigenvalue of $\mathrm{d}/\mathrm{d}x$).

If $g(x) = f(x-a)$ with a constant then

$$G(k) = \int_{-\infty}^{\infty} f(x-a)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(z)\mathrm{e}^{\mathrm{i}k(a+z)}\,\mathrm{d}z = \mathrm{e}^{\mathrm{i}ka}F(k)$$

by putting $x = a+z$. Thus translation becomes multiplication by a phase.

Conversely, if $g(x) = \mathrm{e}^{\mathrm{i}ax}f(x)$ then

$$G(k) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ax}f(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{\mathrm{i}(k+a)x}\,\mathrm{d}x = F(k+a).$$

Theorem 2.1. Convolution Theorem. If

$$h(x) = \int_{-\infty}^{\infty} g(x-y)f(y)\,\mathrm{d}y$$

then

$$H(k) = G(k)F(k).$$

Proof. We can use the shift formula:

$$\begin{aligned} H(k) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x-y)f(y)\,\mathrm{d}y \right) \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} g(x-y)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x \right) f(y)\,\mathrm{d}y \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky}G(k)\,f(y)\,\mathrm{d}y = G(k)F(k). \end{aligned}$$

□

Theorem 2.2. Plancharel Theorem Equivalent to Parseval for sums.

$$\int_{-\infty}^{\infty} |f(x)|^2\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(k)|^2\,\mathrm{d}k.$$

2.3 Examples

Example 2.1. Transform of a delta function.

$$\overline{\delta(x)} = \int_{-\infty}^{\infty} \delta(x)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = 1.$$

The inverse transform is

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x.$$

Similarly

$$\bar{1} = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = 2\pi\delta(k), \qquad 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(k)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}k.$$

■

Example 2.2. Consider $f(x) = \mathrm{e}^{-a|x|}$. The Fourier transform is

$$F(k) = \int_{-\infty}^{0} \mathrm{e}^{ax}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x + \int_{0}^{\infty} \mathrm{e}^{-ax}\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = \left[\frac{\mathrm{e}^{(a+\mathrm{i}k)x}}{a+\mathrm{i}k}\right]_{-\infty}^{0} + \left[\frac{\mathrm{e}^{(-a+\mathrm{i}k)x}}{-a+\mathrm{i}k}\right]_{0}^{\infty}$$
$$= \frac{1}{a+\mathrm{i}k} - \frac{1}{-a+\mathrm{i}k} = \frac{2a}{a^2+k^2}.$$

The inverse transform, for $x > 0$ say, is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2a\mathrm{e}^{-\mathrm{i}kx}}{a^2+k^2}\,\mathrm{d}k.$$

Since $\mathrm{Re}(-\mathrm{i}kx) = \mathrm{Im}(k)x$ for x real, $\mathrm{e}^{-\mathrm{i}kx}$ is exponentially small in the lower half plane if $x > 0$. Close the contour with a large semi-circular arc in the lower half-plane. The contribution from the circular arc tends to zero as the arc tends to infinity (Jordan's lemma). We are left with the residue contribution from the pole at $k = -\mathrm{i}a$, which we are circling clockwise. Thus, for $x > 0$,

$$f(x) = -2\pi\mathrm{i}\frac{1}{2\pi}\frac{2a\mathrm{e}^{-ax}}{-2\mathrm{i}a} = \mathrm{e}^{-ax}.$$

For $x < 0$ close with a semicircular contour in the upper half plane to get

$$f(x) = 2\pi\mathrm{i}\frac{1}{2\pi}\frac{2a\mathrm{e}^{ax}}{2\mathrm{i}a} = \mathrm{e}^{ax}.$$

Together $f(x) = \mathrm{e}^{-a|x|}$ as expected.

■

2.4 Laplace Transform

Suppose we have a function which is only defined for $t \geq 0$. If we do not care about $t \leq 0$ we may set $f = 0$ for $t < 0$. The Fourier transform of f is then

$$F(k) = \int_0^\infty \mathrm{e}^{\mathrm{i}kt} f(t)\,\mathrm{d}t$$

with inverse

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kt} F(k)\,dk.$$

If $F(k)$ exists for real k it is analytic in the upper half k-plane. Now we make the change of variables $k = \mathrm{i}p$ with $p > 0$ we find

$$F(\mathrm{i}p) = \int_0^\infty \mathrm{e}^{-pt} f(t)\,\mathrm{d}t$$

with inverse

$$f(t) = \frac{\mathrm{i}}{2\pi}\int_{\mathrm{i}\infty}^{-\mathrm{i}\infty} \mathrm{e}^{pt} F(\mathrm{i}p)\,\mathrm{d}p = \frac{1}{2\pi \mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathrm{e}^{pt} F(\mathrm{i}p)\,\mathrm{d}p.$$

The quantity $F(\mathrm{i}p)$ is the Laplace transform of f. In fact the Laplace transform also exists for functions which grow at infinity (not too quickly), for which the Fourier transform does not exist (for real k). In general, writing $F(\mathrm{i}p) = \tilde{f}(p)$, we have

$$\tilde{f}(p) = \int_0^\infty \mathrm{e}^{-pt} f(t)\,\mathrm{d}t, \qquad f(t) = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{pt} \tilde{f}(p)\,\mathrm{d}p,$$

where c is such that the inversion contour lies to the right all all sigularities of $\tilde{f}$ in the complex plane. This ensures that for $t < 0$ the contour can be deformed to infinity giving $f = 0$ for $t < 0$.

Combining the transform and its inverse gives

$$\begin{aligned} f(t) &= \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{pt}\left(\int_0^\infty \mathrm{e}^{-p\tau} f(\tau)\,\mathrm{d}\tau\right)\mathrm{d}p \\ &= \int_0^\infty \left(\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{p(t-\tau)}\,\mathrm{d}p\right) f(\tau)\,\mathrm{d}\tau \end{aligned}$$

giving the resolution of the identity as

$$\delta(\tau - t) = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{p(t-\tau)}\,\mathrm{d}p.$$

3 Hankel transforms

3.1 Discrete eigenfunction expansion

3.1.1 Eigenfunctions

Consider axisymmetric eigenfunctions of the 2d Laplacian in plane polar coordinates

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} = \lambda u, \tag{3.1}$$

with $u = 0$ (say) on $r = R$. If we write $\lambda = -k^2$ then the solutions to the equation are the Bessel functions of order zero,

$$J_0(kr) \qquad \text{and} \qquad Y_0(kr).$$

Since Y_0 is unbounded at the origin, we must have

$$u = cJ_0(kr).$$

The eigenvalues are determined by imposing the boundary condition

$$J_0(kR) = 0.$$

This equation has roots k_n where $0 < k_1 < k_2 < k_3 < \cdots$.

The Sturm-Liouville form of (3.1) is

$$Lu = \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) = \lambda ru = -k^2ru.$$

The r on the RHS means that this doesn't quite fit into the framework of §1.2: there is an extra weight function r which needs to be included in the integrals. With $\langle\cdot,\cdot\rangle$ defined as before

$$-k_n^2\langle u_m, ru_n\rangle = \langle u_m, Lu_n\rangle = \langle Lu_m, u_n\rangle = -k_m^2\langle ru_m, u_n\rangle.$$

Thus, if $n \neq m$ then

$$\int_0^R u_m u_n r \,\mathrm{d}r = 0, \qquad \text{i.e.} \qquad \int_0^R J_0(k_m r) J_0(k_n r) r \,\mathrm{d}r = 0.$$

To normalise the eigenfunctions we evaluate

$$\int_0^R J_0(k_n r)^2 r \,\mathrm{d}r = \frac{R^2}{2} J_1(k_n R)^2.$$

Thus

$$u_n = \frac{J_0(k_n r)\sqrt{2}}{R|J_1(k_n R)|},$$

satisfies

$$\int_0^R u_n u_m r \,\mathrm{d}r = \delta_{nm}.$$

We can now expand a given function f in terms of these eigenfunctions as

$$f(r) = \sum_{n=1}^{\infty} c_n u_n(r)$$

where

$$c_n = \int_0^R r f(r) u_n(r) \,\mathrm{d}r.$$

Then

$$f(r) = \sum_{n=1}^{\infty} \left(\int_0^R \rho f(\rho) u_n(\rho) \,\mathrm{d}\rho \right) u_n(r) = \int_0^R \left(\sum_{n=1}^{\infty} \rho u_n(\rho) u_n(r) \right) f(\rho) \,\mathrm{d}\rho,$$

so that

$$\sum_{n=1}^{\infty} \rho u_n(\rho) u_n(r) = \sum_{n=1}^{\infty} \frac{2\rho J_0(k_n \rho) J_0(k_n r)}{R^2 J_1(k_n R)^2} = \delta(\rho - r). \tag{3.2}$$

Note that this implies that

$$\sum_{n=1}^{\infty} r u_n(\rho) u_n(r) = \sum_{n=1}^{\infty} \frac{2r J_0(k_n \rho) J_0(k_n r)}{R^2 J_1(k_n R)^2} = \delta(\rho - r)$$

also.

3.1.2 Inhomogeneous equation

Now we can solve

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} = f, \tag{3.3}$$

with $u = 0$ on $r = R$ via an eigenfunction expansion. Writing

$$u(r) = \sum_{n=1}^{\infty} a_n u_n(r), \qquad f(r) = \sum_{n=1}^{\infty} c_n u_n(r)$$

gives

$$\sum_{n=1}^{\infty} -k_n^2 a_n u_n(r) = \sum_{n=1}^{\infty} c_n u_n(r).$$

Now multiply by $ru_m(r)$ and integrate to give

$$-k_m^2 a_m = c_m \qquad \text{i.e.} \qquad a_m = -\frac{c_m}{k_m^2}.$$

Thus

$$u(r) = \sum_{n=1}^{\infty} -\frac{c_n u_n(r)}{k_n^2}.$$

Inserting our expression for c_n gives

$$\begin{aligned} u(r) &= \sum_{n=1}^{\infty} \left(-\int_0^R \rho f(\rho) u_n(\rho)\,\mathrm{d}\rho \right) \frac{u_n(r)}{k_n^2} \\ &= \int_0^R \sum_{n=1}^{\infty} \left(-\frac{r u_n(r) u_n(\rho)}{k_n^2} \right) f(\rho)\,\mathrm{d}\rho = \int_0^R G(\rho, r) f(\rho)\,\mathrm{d}\rho \end{aligned}$$

where the Green's function

$$G(r,\rho) = \sum_{n=1}^{\infty} -\frac{r u_n(r) u_n(\rho)}{k_n^2} = \sum_{n=1}^{\infty} \frac{r u_n(r) u_n(\rho)}{\lambda_n} = \sum_{n=1}^{\infty} -\frac{2r J_0(k_n r) J_0(k_n \rho)}{k_n^2 R^2 |J_1(k_n R)|^2}.$$

3.2 Hankel transform

Instead of a finite disc, now consider an infinite domain. We replace the discrete spectrum of eigenfunctions $J_0(k_n r)$ by a continuous spectrum $J_0(kr)$ with continuous eigenvalue k.

The resolution of the identity is

$$\int_0^\infty kr J_0(kr) J_0(k\rho)\,\mathrm{d}k = \delta(r-\rho). \tag{3.4}$$

This is the continuous version of (3.2). This resolution of the identity tells us what the transform pair is. The Hankel transform is

$$\hat{u}(k) = \int_0^\infty r u(r) J_0(kr)\,\mathrm{d}r$$

with the inversion given by

$$u(r) = \int_0^\infty k\hat{u}(k) J_0(kr)\,\mathrm{d}k.$$

Together we have

$$u(\rho) = \int_0^\infty \left(\int_0^\infty r u(r) J_0(kr)\,\mathrm{d}r\right) k J_0(k\rho)\,\mathrm{d}k = \int_0^\infty \left(\int_0^\infty kr J_0(kr) J_0(k\rho)\,\mathrm{d}k\right) u(r)\,\mathrm{d}r$$

giving (3.4).

To solve the inhomogeneous equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} = f, \tag{3.5}$$

with u decaying at infinity, apply the Hankel transform. On the left-hand side this gives

$$\begin{aligned}
\int_0^\infty \left(\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r}\right) r J_0(kr)\,\mathrm{d}r &= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) J_0(kr)\,\mathrm{d}r \\
&= -\int_0^\infty r\frac{\mathrm{d}u}{\mathrm{d}r}\frac{\mathrm{d}}{\mathrm{d}r} J_0(kr)\,\mathrm{d}r \\
&= \int_0^\infty u\left(\frac{\mathrm{d}}{\mathrm{d}r}J_0(kr) + r\frac{\mathrm{d}^2}{\mathrm{d}r^2}J_0(kr)\right)\,\mathrm{d}r \\
&= -\int_0^\infty k^2 u r J_0(kr)\,\mathrm{d}r = -k^2\hat{u}(k)
\end{aligned}$$

on integration by parts, where we have used the fact that $J_0(kr)$ satisfies (3.1):

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}J_0(kr) + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}J_0(kr) = -k^2 J_0(kr). \tag{3.6}$$

This is why the Hankel transform is the natural transform for this operator. Thus the transformed equation is

$$-k^2\hat{u}(k) = \hat{f}$$

i.e.

$$\hat{u}(k) = \frac{\hat{f}}{-k^2}.$$

Compare to (2.2). Inverting gives

$$u(\rho) = \int_0^\infty \frac{\hat{f} J_0(k\rho)}{-k}\,\mathrm{d}k.$$

Inserting the expression for $\hat{f}$ we can try and find the Green's function

$$\begin{aligned} u(\rho) &= \int_0^\infty \frac{\hat{f}}{-k^2} k J_0(k\rho)\,\mathrm{d}k = \int_0^\infty \left(\int_0^\infty r u(r) J_0(kr)\,\mathrm{d}r \right) \frac{k J_0(k\rho)}{-k^2}\,\mathrm{d}k \\ &= \int_0^\infty \left(\int_0^\infty \frac{r k J_0(kr) J_0(k\rho)}{-k^2}\,\mathrm{d}k \right) u(r)\,\mathrm{d}r \\ &= \int_0^\infty G(\rho, r) u(r)\,\mathrm{d}r. \end{aligned}$$

where

$$G(\rho, r) = \int_0^\infty \frac{r k J_0(kr) J_0(k\rho)}{-k^2}\,\mathrm{d}k = \int_0^\infty \frac{r k J_0(kr) J_0(k\rho)}{\lambda(k)}\,\mathrm{d}k. \tag{3.7}$$

Unfortunately in this case this doesn't quite work because the resulting integral doesn't exist. This is associated with the fact the Green's function behaves like $\log r$ at infinity (see example 3.2).

3.3 Examples

Example 3.1. Green's function for modified Helmholtz eqn.

The Green's function for the modified Helmholtz equation in 2d satisfies

$$\nabla^2 G - \alpha^2 G = -\delta(x)\delta(y)$$

with G decaying at infinity. The solution is axisymmetric, giving

$$\frac{\mathrm{d}^2 G}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}G}{\mathrm{d}r} - \alpha^2 G = -\frac{\delta_r(r)}{2\pi r},$$

where $\delta_r(r)$ is the half-range radial δ-function:

$$1 = \int\int \delta(x)\delta(y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi} \int_0^\infty \frac{\delta_r(r)}{2\pi r} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_0^\infty \delta_r(r)\,\mathrm{d}r.$$

Taking a Hankel transform gives

$$-k^2\hat{G} - \alpha^2\hat{G} = -\int_0^\infty \frac{\delta_r(r)}{2\pi r} r J_0(kr)\,\mathrm{d}r = -\frac{J_0(0)}{2\pi} = -\frac{1}{2\pi}.$$

Thus

$$\hat{G} = \frac{1}{2\pi(k^2+\alpha^2)}.$$

Inverting gives

$$G = \frac{1}{2\pi}\int_0^\infty \frac{kJ_0(kr)}{k^2+\alpha^2}\,\mathrm{d}k = \frac{1}{2\pi}K_0(\alpha r). \tag{3.8}$$

The evaluation of the integral is via contour deformation with residues at $k = \pm\mathrm{i}\alpha$.

■

Example 3.2. Green's function for Laplace eqn. If $\alpha = 0$ we have instead Laplace's eqn. Unfortunately we cannot apply the Hankel transform directly in this case because G grows at infinity.

We can try to take the limit $\alpha \to 0$ in the previous example. However the integral in (3.8) fails to exist when $\alpha = 0$ [of course we can take the limit $\alpha \to 0$ in $K_0(\alpha r)$].

One way out is to differentiate with respect to r to give

$$\frac{\mathrm{d}G}{\mathrm{d}r} = \frac{1}{2\pi}\int_0^\infty \frac{k^2 J_0'(kr)}{k^2+\alpha^2}\,\mathrm{d}k.$$

Now we can safely let $\alpha \to 0$ to give

$$\frac{\mathrm{d}G}{\mathrm{d}r} = \frac{1}{2\pi}\int_0^\infty J_0'(kr)\,\mathrm{d}k = \frac{1}{2\pi}\left[\frac{J_0(kr)}{r}\right]_0^\infty = -\frac{1}{2\pi r}.$$

Thus

$$G = -\frac{1}{2\pi}\log r$$

as expected.

■

3.4 Direct proof of inversion formula

Suppose $f(\rho)$ is analytic in some region containing $[a, b]$. Consider

$$I(t) = \int_0^\infty \left(\int_a^b z f(z) J_0(kz)\,\mathrm{d}z\right) kJ_0(kt)\,\mathrm{d}k.$$

The part in parentheses is the forward Hankel transform of the function

$$\begin{cases} f(z) & a < z < b, \\ 0 & \text{otherwise} \end{cases}$$

To prove the inversion formula we need to show that I is equal this function. Recall the Hankel functions

$$H_0^{(1)}(kz) = J_0(kz) + \mathrm{i}Y_0(kz), \qquad H_0^{(2)}(kz) = J_0(kz) - \mathrm{i}Y_0(kz).$$

$H_0^{(1)}$ decays exponentially in the upper half plane, $H_0^{(2)}$ decays exponentially in the lower half plane. We write

$$J_0(kz) = \frac{1}{2}\left(H_0^{(1)}(kz) + H_0^{(2)}(kz)\right),$$

giving

$$I(t) = \frac{1}{2}\int_0^\infty \int_{C_+} zf(z)H_0^{(1)}(kz)\,\mathrm{d}z\,kJ_0(kt)\,\mathrm{d}k + \frac{1}{2}\int_0^\infty \int_{C_-} zf(z)H_0^{(2)}(kz)\,\mathrm{d}z\,kJ_0(kt)\,\mathrm{d}k.$$

Use the decay of H_0 to swap the order of integration

$$\begin{aligned} I(t) &= \frac{1}{2}\int_{C_+}\int_0^\infty zf(z)H_0^{(1)}(kz)kJ_0(kt)\,\mathrm{d}k\,\mathrm{d}z\, + \frac{1}{2}\int_{C_-}\int_0^\infty zf(z)H_0^{(2)}(kz)kJ_0(kt)\,\mathrm{d}k\,\mathrm{d}z \\ &= I_1(t) + I_2(t), \end{aligned}$$

say. Now (compare (3.6))

$$\frac{\mathrm{d}^2}{\mathrm{d}k^2}J_0(kt) + \frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}k}J_0(kt) = -t^2 J_0(kt),$$

$$\frac{\mathrm{d}^2}{\mathrm{d}k^2}H_0^{(\cdot)}(kz) + \frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}k}H_0^{(\cdot)}(kz) = -z^2 H_0^{(\cdot)}(kz).$$

Thus

$$\begin{aligned} t^2\int_A^B kH_0^{(1)}(kz)J_0(kt)\,\mathrm{d}k &= -\int_A^B kH_0^{(1)}(kz)\left(\frac{\mathrm{d}^2}{\mathrm{d}k^2}J_0(kt) + \frac{1}{k}\frac{\mathrm{d}}{\mathrm{d}k}J_0(kt)\right)\,\mathrm{d}k \\ &= -\int_A^B H_0^{(1)}(kz)\frac{\mathrm{d}}{\mathrm{d}k}\left(k\frac{\mathrm{d}}{\mathrm{d}k}J_0(kt)\right)\,\mathrm{d}k \\ &= -\left[H_0^{(1)}(kz)\left(k\frac{\mathrm{d}}{\mathrm{d}k}J_0(kt)\right)\right]_A^B + \int_A^B \frac{\mathrm{d}}{\mathrm{d}k}H_0^{(1)}(kz)\,k\frac{\mathrm{d}}{\mathrm{d}k}J_0(kt)\,\mathrm{d}k \\ &= \left[kJ_0(kt)\frac{\mathrm{d}}{\mathrm{d}k}H_0^{(1)}(kz) - kH_0^{(1)}(kz)\frac{\mathrm{d}}{\mathrm{d}k}J_0(kt)\right]_A^B - \int_A^B \frac{\mathrm{d}}{\mathrm{d}k}\left(k\frac{\mathrm{d}}{\mathrm{d}k}H_0^{(1)}(kz)\right)J_0(kt)\,\mathrm{d}k \\ &= \left[kJ_0(kt)\frac{\mathrm{d}}{\mathrm{d}k}H_0^{(1)}(kz) - kH_0^{(1)}(kz)\frac{\mathrm{d}}{\mathrm{d}k}J_0(kt)\right]_A^B + z^2\int_A^B kH_0^{(1)}(kz)J_0(kt)\,\mathrm{d}k. \end{aligned}$$

Thus

$$(t^2 - z^2) \int_A^B k H_0^{(1)}(kz) J_0(kt)\,\mathrm{d}k = \Big[kz J_0(kt) H_0^{(1)\prime}(kz) - kt H_0^{(1)}(kz) J_0'(kt) \Big]_A^B .$$

Let $A \to 0$ and $B \to \infty$ to find

$$(t^2 - z^2) \int_0^\infty k H_0^{(1)}(kz) J_0(kt)\,\mathrm{d}k = \lim_{k \to 0} \Big(kt H_0^{(1)}(kz) J_0'(kt) - kz J_0(kt) H_0^{(1)\prime}(kz) \Big) .$$

Now

$$J_0(kt) \to 1, \quad J_0'(kt) \to 0 \qquad \text{as } k \to 0$$

but

$$H_0^{(1)}(kz) \sim \frac{2\mathrm{i}}{\pi} \log(kz) \quad \text{as } k \to 0$$

so that

$$H_0^{(1)\prime}(kz) \sim \frac{2\mathrm{i}}{\pi k z} \quad \text{as } k \to 0$$

Thus

$$(t^2 - z^2) \int_0^\infty k H_0^{(1)}(kz) J_0(kt)\,\mathrm{d}k = \lim_{k \to 0} -kz \frac{2\mathrm{i}}{\pi k z} = -\frac{2\mathrm{i}}{\pi}.$$

Similarly

$$(t^2 - z^2) \int_0^\infty k H_0^{(2)}(kz) J_0(kt)\,\mathrm{d}k = \frac{2\mathrm{i}}{\pi}.$$

Therefore

$$\begin{aligned} I(t) &= \frac{\mathrm{i}}{\pi} \int_{C_+} \frac{z f(z)}{z^2 - t^2}\,\mathrm{d}z - \frac{\mathrm{i}}{\pi} \int_{C_-} \frac{z f(z)}{z^2 - t^2}\,\mathrm{d}z \\ &= -\frac{\mathrm{i}}{\pi} \oint_{C_- - C_+} \frac{z f(z)}{z^2 - t^2}\,\mathrm{d}z \\ &= \frac{1}{2\pi\mathrm{i}} \oint_{C_- - C_+} f(z) \left(\frac{1}{z-t} + \frac{1}{z+t} \right) \mathrm{d}z \\ &= \begin{cases} f(t) & a < t < b, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

as required.

3.5 Hankel transform as axisymmetric Fourier transform

Suppose we have a function $U(x, y)$ of two variables which depends only on $r = \sqrt{x^2 + y^2}$, so that $U(x, y) = u(r)$ say. If we take the Fourier transform in the two variables x and

y then, with $\mathbf{x} = (x, y)$ and $\mathbf{k} = (k_1, k_2)$,

$$\tilde{U}(k_1, k_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x, y)\mathrm{e}^{\mathrm{i}k_1 x}\mathrm{e}^{\mathrm{i}k_2 y}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\int_0^{\infty} \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}u(r)r\,\mathrm{d}r\,\mathrm{d}\theta$$
$$= \int_0^{2\pi}\int_0^{\infty} \mathrm{e}^{\mathrm{i}kr\cos\theta}u(r)r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi\int_0^{\infty} ru(r)J_0(kr)\,\mathrm{d}r = 2\pi\hat{u}(k)$$

where θ is the angle between $\mathbf{x}$ and $\mathbf{k}$, and $k = |\mathbf{k}|$. Here we have used the integral representation of the Bessel function

$$J_0(kr) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}kr\cos\theta}\,\mathrm{d}\theta.$$

Inversion is given by

$$U(\mathbf{x}) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\tilde{U}(\mathbf{k})\,\mathrm{d}k_1\,\mathrm{d}k_2$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} \mathrm{e}^{-\mathrm{i}kr\cos\theta}\hat{u}(k)k\,\mathrm{d}\theta\,\mathrm{d}k$$
$$= \int_0^{\infty}\left(\frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kr\cos\theta}\,\mathrm{d}\theta\right)\hat{u}(k)k\,\mathrm{d}k$$
$$= \int_0^{\infty} J_0(kr)\hat{u}(k)k\,\mathrm{d}k = u(r).$$

Aside

The integral representation of the Bessel function can be derived from the generating function

$$\phi = \mathrm{e}^{z/2(t-1/t)} = \sum_{n=-\infty}^{\infty} t^n J_n(z).$$

$J_0(z)$ is the coefficient of t^0 in the expansion of ϕ about $t = 0$. We can find this from the Cauchy integral:

$$J_0(z) = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{\phi}{t}\,\mathrm{d}t$$

where the integration is around the unit circle in the complex plane. Thus, using the parametrisation $t = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}$ for the unit circle,

$$J_0(z) = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{\mathrm{e}^{z/2(t-1/t)}}{t}\,\mathrm{d}t = \frac{1}{2\pi\mathrm{i}}\int_0^{2\pi} \frac{\mathrm{e}^{z/2(\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}-1/\mathrm{i}\mathrm{e}^{-\mathrm{i}\theta})}}{\mathrm{i}\mathrm{e}^{\mathrm{i}\theta}}(-\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}z\cos\theta}\,\mathrm{d}\theta.$$

3.6 Parseval

$$\int_0^\infty k\hat{u}(k)\hat{v}(k)\,\mathrm{d}k = \int_0^\infty k\hat{u}(k)\int_0^\infty rv(r)J_0(kr)\,\mathrm{d}r\,\mathrm{d}k$$
$$= \int_0^\infty rv(r)\int_0^\infty k\hat{u}(k)J_0(kr)\,\mathrm{d}k\,\mathrm{d}r = \int_0^\infty rv(r)u(r)\,\mathrm{d}r.$$

3.7 Applications

Example 3.3. Point charge between two parallel plates Consider the following Poisson equation in three dimensions:

$$\nabla^2\phi = -4\pi\delta(x)\delta(y)\delta(z)$$

with $\phi = 0$ on $z = \pm a$. As in Example 3.1 we look for an axisymmetric solution $\phi = \phi(r, z)$, giving

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial z^2} = -4\pi\frac{\delta_r(r)}{2\pi r}\delta(z) = -2\delta_r(r)\delta(z).$$

Now take the Hankel transform to give

$$-k^2\hat{\phi} + \frac{\partial^2\hat{\phi}}{\partial z^2} = -2\delta(z),$$

with $\hat{\phi} = 0$ on $z = \pm a$. Thus $\hat{\phi}$ is the Green's function for the one-dimensional modified Helmholtz equation. The solution is

$$\hat{\phi} = \frac{\sinh(k(a-|z|))}{k\cosh(ka)} = \begin{cases} \dfrac{\sinh(k(z+a))}{k\cosh(ka)} & \text{if } z < 0, \\ \dfrac{\sinh(k(a-z))}{k\cosh(ka)} & \text{if } z > 0. \end{cases}$$

Can easily check that this satisfies the equation for $z \neq 0$ and that

$$\left[\hat{\phi}\right]_{z=0-}^{z=0+} = 0, \qquad \left[\frac{\partial\hat{\phi}}{\partial z}\right]_{z=0-}^{z=0+} = -2.$$

We can rewrite

$$\begin{aligned}\hat{\phi} &= \frac{e^{ka}e^{-k|z|} - e^{-ka}e^{k|z|}}{2k\cosh(ka)} = \frac{(e^{ka} + e^{-ka})e^{-k|z|} - e^{-ka}(e^{-k|z|} + e^{k|z|})}{2k\cosh(ka)} \\ &= \frac{\cosh(ka)e^{-k|z|} - e^{-ka}\cosh(k|z|)}{k\cosh(ka)} \\ &= \frac{e^{-k|z|}}{k} - \frac{\cosh(k|z|)}{\cosh(ka)}\frac{e^{-ka}}{k}.\end{aligned}$$

Thus

$$\phi(r, z) = \int_0^\infty e^{-k|z|} J_0(kr)\,dk - \int_0^\infty e^{-ka}\frac{\cosh(k|z|)}{\cosh(ka)} J_0(kr)\,dk.$$

In fact the first term corresponds to the solution for a point charge in free space (note it is independent of a),

$$\int_0^\infty e^{-k|z|} J_0(kr)\,dk = \frac{1}{\sqrt{r^2 + z^2}}.$$

Note that this is the Laplace transform in k of $J_0(kr)$.

■

4 Mellin Transforms

Laplace's equation in cylindrical polars is

$$\nabla^2 U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0.$$

Seek a separable solution $U(r,\theta) = u(r)\mathrm{e}^{\mathrm{i}k\theta}$ giving

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{k^2}{r^2}u = 0,$$

which we may write

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) = k^2 u.$$

This is homogeneous in r with solutions $r^{\pm k}$.

The Mellin transform is

$$\bar{u}(s) = \int_0^\infty r^{s-1}u(r)\,\mathrm{d}r,$$

defined for those complex s for which the integral converges. Suppose

$$u(r) = \begin{cases} O(r^{-\alpha}) & \text{as } r \to 0, \\ O(r^{-\beta}) & \text{as } r \to \infty. \end{cases}$$

Then $\bar{u}(s)$ exists and is analytic for s in the strip $\alpha < \mathrm{Re}(s) < \beta$.

The inversion formula is

$$u(r) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} r^{-s}\bar{u}(s)\,\mathrm{d}s,$$

where $\alpha < c < \beta$.

Combining the transform and its inverse gives

$$\begin{aligned} u(r) &= \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} r^{-s}\int_0^\infty \rho^{s-1}u(\rho)\,\mathrm{d}\rho\,\mathrm{d}s \\ &= \int_0^\infty \left(\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} r^{-s}\rho^{s-1}\,\mathrm{d}s\right)u(\rho)\,\mathrm{d}\rho \end{aligned}$$

so that the resolution of the identity is

$$\delta(r-\rho) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} r^{-s} \rho^{s-1} \, \mathrm{d}s.$$

To show this put $s = \mathrm{i}\mu$ to give

$$\begin{aligned} \delta(r-\rho) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} r^{-\mathrm{i}\mu} \rho^{\mathrm{i}\mu-1} \, \mathrm{d}\mu \\ &= \frac{1}{2\pi\rho} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu \log(\rho/r)} \, \mathrm{d}\mu \\ &= \frac{1}{\rho} \delta(\log(\rho/r)) \\ &= \delta(r-\rho), \end{aligned}$$

since

$$|f'(\rho)| \, \delta(f(r) - f(\rho)) = \delta(r-\rho).$$

4.1 Relation with Fourier transform

Putting $r = \mathrm{e}^x$ gives

$$r\frac{\mathrm{d}}{\mathrm{d}r} = \mathrm{e}^x \frac{\mathrm{d}x}{\mathrm{d}r} \frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}$$

and thus transforms eigenfunctions of $r\mathrm{d}/\mathrm{d}r$ to eigenfunctions of $\mathrm{d}/\mathrm{d}x$. The domain $r \in [0, \infty)$ maps to $x \in (-\infty, \infty)$.

$$\begin{aligned} \bar{u}(\mathrm{i}\mu) &= \int_0^{\infty} r^{\mathrm{i}\mu-1} u(r) \, \mathrm{d}r \\ &= \int_{-\infty}^{\infty} (\mathrm{e}^x)^{\mathrm{i}\mu-1} u(\mathrm{e}^x) \frac{\mathrm{d}r}{\mathrm{d}x} \, \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu x} u(\mathrm{e}^x) \, \mathrm{d}x \end{aligned}$$

which is the Fourier transform of $u(\mathrm{e}^x)$. Inversion is

$$\begin{aligned} u(\mathrm{e}^x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\mu x} \bar{u}(\mathrm{i}\mu) \, \mathrm{d}\mu \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathrm{e}^x)^{-\mathrm{i}\mu} \bar{u}(\mathrm{i}\mu) \, \mathrm{d}\mu \\ &= \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} (\mathrm{e}^x)^{-s} \bar{u}(s) \, \mathrm{d}s. \end{aligned}$$

4.2 Properties of Mellin transform

With

$$M[f] = \int_0^\infty r^{s-1} f(r)\,\mathrm{d}r \ = F(s)$$

then

$$\begin{aligned} M[f'] &= \int_0^\infty r^{s-1} f'(r)\,\mathrm{d}r \\ &= \left[r^{s-1} f(r)\right]_0^\infty - (s-1)\int_0^\infty r^{s-2} f(r)\,\mathrm{d}r \\ &= -(s-1)F(s-1) \end{aligned}$$

with $\alpha < \mathrm{Re}(s-1) < \beta$ (to discard $[r^{s-1}f(r)]_0^\infty$). Also

$$M\left[r^\mu f(r)\right] = \int_0^\infty r^{s-1} r^\mu f(r)\,\mathrm{d}r = F(s+\mu).$$

4.3 Mellin transform convolutions

If

$$k(r) = \int_0^\infty y^\mu f(r/y) g(y)\,\mathrm{d}y$$

then

$$\begin{aligned} \bar{k}(s) &= \int_0^\infty r^{s-1} \int_0^\infty y^\mu f(r/y) g(y)\,\mathrm{d}y\,\mathrm{d}r \\ &= \int_0^\infty \int_0^\infty r^{s-1} y^\mu f(r/y) g(y)\,\mathrm{d}r\,\mathrm{d}y \\ &= \int_0^\infty \int_0^\infty (yt)^{s-1} y^\mu f(t) g(y) y\,\mathrm{d}t\,\mathrm{d}y \\ &= \int_0^\infty t^{s-1} f(t)\,\mathrm{d}t \int_0^\infty y^{s+\mu} g(y)\,\mathrm{d}y \\ &= \bar{f}(s)\bar{g}(s+\mu+1), \end{aligned}$$

on setting $r = yt$.

Similarly, if

$$h(r) = \int_0^\infty y^\mu f(ry) g(y)\,\mathrm{d}y$$

then

$$\bar{h}(s) = \bar{f}(s)\bar{g}(\mu+1-s).$$

4.4 Solving inhomogeneous equation

To solve

$$r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) = r^2\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + r\frac{\mathrm{d}u}{\mathrm{d}r} = f(r)$$

apply the Mellin transform to give

$$s(s+1)\bar{u}(s) - s\bar{u}(s) = \bar{f}(s),$$

i.e.

$$\bar{u}(s) = \frac{\bar{f}(s)}{s^2}.$$

Now invert to give

$$\begin{aligned}
u(r) &= \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} r^{-s}\frac{\bar{f}(s)}{s^2}\,\mathrm{d}s \\
&= \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} r^{-s}\frac{1}{s^2}\int_0^\infty \rho^{s-1}u(\rho)\,\mathrm{d}\rho\,\mathrm{d}s \\
&= \int_0^\infty \left(\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{r^{-s}\rho^{s-1}}{s^2}\,\mathrm{d}s\right) u(\rho)\,\mathrm{d}\rho \\
&= \int_0^\infty G(r,\rho)u(\rho)\,\mathrm{d}\rho
\end{aligned}$$

where the Green's function

$$G(r,\rho) = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{r^{-s}\rho^{s-1}}{s^2}\,\mathrm{d}s = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{r^{-s}\rho^{s-1}}{\lambda(s)}\,\mathrm{d}s.$$

4.5 Applications

4.5.1 Summation of Series

Let S represent a series of the form

$$S = \sum_{n=1}^{\infty} f(n) \tag{4.1}$$

in which the terms are samples of a function $f(t)$ for integer values of the variable $t \in (0,\infty)$. Suppose the Mellin transform $F(s)$ of $f(t)$ exists in the strip $c_1 < \mathrm{Re}(s) < c_2$. Then, the Mellin inversion formula gives

$$f(t) = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} F(s)t^{-s}\,\mathrm{d}s, \qquad c_1 < c < c_2.$$

Substituting this into (4.1) gives

$$S = \frac{1}{2\pi \mathrm{i}} \sum_{n=1}^{\infty} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} F(s) n^{-s} \, \mathrm{d}s.$$

Now, if $F(s)$ is such that the sum and integral can be exchanged, an integral expression for S is obtained:

$$S = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} F(s) \zeta(s) \, \mathrm{d}s,$$

where $\zeta(s)$ is the Riemann zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}.$$

The integral can then be evaluated via the calculus of residues, which may give an infinite sum which, with luck, will be more rapidly convergent that the original series.

Example 4.1. Compute the sum

$$S(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}.$$

Since

$$\begin{aligned}
\mathcal{M}\left[\mathrm{e}^{\mathrm{i}at}\right] &= \int_0^{\infty} t^{s-1} \mathrm{e}^{\mathrm{i}at} \, \mathrm{d}t \\
&= (-\mathrm{i}a)^{-s} \int_0^{\infty} (-\mathrm{i}at)^{s-1} \mathrm{e}^{\mathrm{i}at} \, (-\mathrm{i}a) \mathrm{d}t \\
&= (-\mathrm{i}a)^{-s} \int_0^{\infty} v^{s-1} \mathrm{e}^{-v} \, \mathrm{d}v \\
&= a^{-s} \mathrm{e}^{\mathrm{i}\pi s/2} \Gamma(s) \qquad 0 < \mathrm{Re}(s) < 1,
\end{aligned}$$

we have

$$\mathcal{M}\left[\cos(xt)\right] = \frac{1}{2\mathrm{i}} \left(x^{-s} \Gamma(s) \mathrm{e}^{\mathrm{i}\pi s/2} + x^{-s} \Gamma(s) \mathrm{e}^{-\mathrm{i}\pi s/2} \right) = x^{-s} \Gamma(s) \cos\left(\frac{\pi s}{2}\right) \qquad 0 < \mathrm{Re}(s) < 1.$$

Thus

$$\mathcal{M}\left[\frac{\cos tx}{t^2}\right] = x^{2-s} \Gamma(s-2) \cos\left(\frac{\pi(s-2)}{2}\right) = -x^{2-s} \Gamma(s-2) \cos\left(\frac{\pi s}{2}\right), \qquad 2 < \mathrm{Re}(s) < 3.$$

Hence the sum can be rewritten

$$S = -\frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{2-s} \Gamma(s-2) \cos\left(\frac{\pi s}{2}\right) \zeta(s) \ \ ds,$$

where the interchange of integration and summation is justified by absolute convergence.

To evaluate the integral we use Riemann's functional relationship for the zeta function:

$$\pi^2\zeta(1-s) = 2^{1-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s).$$

We find

$$\begin{aligned} S &= -\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{2-s}2^{s-1}\pi^s\zeta(1-s)\frac{\Gamma(s-2)}{\Gamma(s)}\,\mathrm{d}s \\ &= -\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} x^{2-s}2^{s-1}\pi^s\frac{\zeta(1-s)}{(s-1)(s-2)}\,\mathrm{d}s. \end{aligned}$$

Now we can close the contour in the left half plane and evaluate by a sum of residues. The zeta function $\zeta(s)$ is analytic except for a simple pole at $s=1$ with residue 1. Thus the integrand has poles at $s=1$, $s=2$ and $s=0$ giving

$$\begin{aligned} S &= -\left(\frac{x^2 2^{-1}}{(-1)(-2)} + x\pi\frac{\zeta(0)}{(-1)} + 2\pi^2\frac{\zeta(-1)}{(1)}\right) \\ &= \frac{x^2}{4} - \frac{x\pi}{2} + \frac{\pi^2}{6}, \end{aligned}$$

since $\zeta(0) = -1/2$ and $\zeta(-1) = -1/12$.

■

Note that the pole at $s=m$ produces a term proportional to x^{2-m}. As we move the contour leftwards the powers of x produced are getting larger.

Often the number of poles is infinite, but we can generate an asymptotic series as $x \to 0$ by considering only the first few poles.

4.5.2 Asymptotic evaluation of harmonic sums

A *harmonic sum* is a sum of the form

$$G(x) = \sum_{k=1}^{\infty} \lambda_k g(\mu_k x).$$

Since $\mathcal{M}[f(ax); s] = \mathcal{M}[f(x); s]/a^s = \bar{f}(s)/a^s$ say (see problem sheet), taking a Mellin transform gives

$$\bar{G}(s) = \sum_{k=1}^{\infty} \lambda_k \mu_k^{-s}\bar{g}(s) = \Lambda(s)\bar{g}(s),$$

where

$$\Lambda(s) = \sum_{k=1}^{\infty} \lambda_k \mu_k^{-s}.$$

Thus the "amplitude-frequency" information λ_k, μ_k is separated from the "base function" $g(x)$.

Example 4.2. Find the asumptotic behaviour of

$$G(x) = \sum_{k=1}^{\infty} \mathrm{e}^{-k^2 x}$$

as $x \to 0$.

Here $\lambda_k = 1$, $\mu_k = k^2$, $g(x) = \mathrm{e}^{-x}$. Since

$$\mathcal{M}[\mathrm{e}^{-x}, s] = \int_0^{\infty} x^{s-1} \mathrm{e}^{-x} \, \mathrm{d}x = \Gamma(s),$$

(where $\mathrm{Re}(s) > 0$) taking a Mellin transform gives

$$\bar{G}(s) = \Lambda(s)\Gamma(s),$$

where

$$\Lambda(s) = \sum_{k=1}^{\infty} (k^2)^{-s} = \zeta(2s),$$

providing $\mathrm{Re}(s) > 1/2$. Hence

$$G(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \zeta(2s)\Gamma(s) x^{-s} \, \mathrm{d}s,$$

where $c > 1/2$. Now $\zeta(s)$ has a pole at $s = 1$ with residue 1, while $\Gamma(s)$ has poles at the negative integers $s = -m$, $m = 0, 1, 2, \ldots$, with residue $(-1)^m/m!$. Note that $\zeta(-2m) = 0$ for $m = 1, 2, \ldots$. Moving the contour to the left (i.e. reducing c so that $c < 0$) gives

$$\begin{aligned} G(x) &= \mathrm{res}(\zeta(2s)\Gamma(s)x^{-s} | s = 1/2) + \mathrm{res}(\zeta(2s)\Gamma(s)x^{-s} | s = 0) + \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \zeta(2s)\Gamma(s)x^{-s} \, \mathrm{d}s \\ &= \frac{\Gamma(1/2)}{2x^{1/2}} + \zeta(0) + O(x^{-c}) \\ &= \frac{\sqrt{\pi}}{2\sqrt{x}} - \frac{1}{2} + O(x^{-c}) \end{aligned}$$

for any $c < 0$ (thus the error is exponentially small).

■

Example 4.3. Let us revisit Example 4.1. We have $\lambda_k = 1/k^2$, $\mu_k = k$, $g(x) = \cos(x)$. We have

$$\mathcal{M}[\cos x; s] = \Gamma(s)\cos\left(\frac{\pi s}{2}\right)$$

if $\mathrm{Re}(s) > 0$, while

$$\Lambda(s) = \sum_{k=1}^{\infty} \frac{1}{k^2}\frac{1}{k^s} = \sum_{k=1}^{\infty} \frac{1}{k^{s+2}} = \zeta(s+2),$$

providing $\mathrm{Re}(s) > -1$. Hence

$$\bar{S}(s) = \zeta(s+2)\Gamma(s)\cos\left(\frac{\pi s}{2}\right).$$

Thus

$$S(x) = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \zeta(s+2)\Gamma(s)\cos\left(\frac{\pi s}{2}\right) x^{-s}\,\mathrm{d}s,$$

which of course is exactly what we found before. If instead of using the functional relation for the zeta function we just shift the contour leftwards we will pick up the asymptotic expansion of the sum as $x \to 0$. There is a pole at $s = -1$ due to $\zeta(s+2)$, and poles at $s = -m$, $m = 0, 1, 2, \ldots$ due to $\Gamma(s)$. But at the even integers $\zeta(-2m) = 0$, while at the odd integers $\cos(\pi/2(2m+1)) = 0$. Thus we only have residue contributions from $s = -1$, $s = 0$, and $s = -2$. Evaluating the sum of residues gives

$$S = \frac{\pi^2}{6} - \frac{x\pi}{2} + \frac{x^2}{4} + O(x^b),$$

for any $b > 2$, i.e. the error is again exponentially small. Of course, in this case we know the error is identically zero.

■

4.5.3 Potential problem in a wedge

Let ϕ satisfy the two-dimensional Laplace equation in the wedge $0 < r < \infty$, $-\alpha < \theta < \alpha$ with boundary conditions

$$\phi(r) = f_{\pm}(r) \quad \text{on } \theta = \pm\alpha,$$

with ϕ bounded for finite r and $\phi \to 0$ as $r \to \infty$. In polar coordinates Laplace's equation is

$$r^2\frac{\partial^2\phi}{\partial r^2} + r\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial\theta^2} = 0.$$

Taking the Mellin transform gives

$$s^2\Phi(s,\theta) + \frac{\mathrm{d}^2\Phi}{\mathrm{d}\theta^2}(s,\theta) = 0.$$

The general solution is

$$\Phi(s,\theta) = A(s)\mathrm{e}^{\mathrm{i}s\theta} + B(s)\mathrm{e}^{-\mathrm{i}s\theta}.$$

The boundary conditions give

$$\begin{aligned} A(s)\mathrm{e}^{-\mathrm{i}s\alpha} + B(s)\mathrm{e}^{\mathrm{i}s\alpha} &= F_-(s), \\ A(s)\mathrm{e}^{\mathrm{i}s\alpha} + B(s)\mathrm{e}^{-\mathrm{i}s\alpha} &= F_+(s), \end{aligned}$$

where $F_\pm(s)$ are the Mellin transforms of $f_\pm(r)$. Solving for A and B gives

$$\begin{aligned} A(s) &= \frac{\mathrm{e}^{\mathrm{i}s\alpha}F_+(s) - \mathrm{e}^{-\mathrm{i}s\alpha}F_-(s)}{\mathrm{e}^{2\mathrm{i}s\alpha} - \mathrm{e}^{-2\mathrm{i}s\alpha}} = \frac{\mathrm{e}^{\mathrm{i}s\alpha}F_+(s) - \mathrm{e}^{-\mathrm{i}s\alpha}F_-(s)}{2\mathrm{i}\sin(2s\alpha)}, \\ B(s) &= \frac{\mathrm{e}^{\mathrm{i}s\alpha}F_-(s) - \mathrm{e}^{-\mathrm{i}s\alpha}F_+(s)}{\mathrm{e}^{2\mathrm{i}s\alpha} - \mathrm{e}^{-2\mathrm{i}s\alpha}} = \frac{\mathrm{e}^{\mathrm{i}s\alpha}F_-(s) - \mathrm{e}^{-\mathrm{i}s\alpha}F_+(s)}{2\mathrm{i}\sin(2s\alpha)}. \end{aligned}$$

Thus

$$\begin{aligned} \phi(r,\theta) &= \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \left(\frac{\mathrm{e}^{\mathrm{i}s(\theta+\alpha)}F_+(s) - \mathrm{e}^{\mathrm{i}s(\theta-\alpha)}F_-(s)}{2\mathrm{i}\sin(2s\alpha)} + \frac{\mathrm{e}^{-\mathrm{i}s(\theta-\alpha)}F_-(s) - \mathrm{e}^{-\mathrm{i}s(\theta+\alpha)}F_+(s)}{2\mathrm{i}\sin(2s\alpha)} \right) r^{-s}\,\mathrm{d}s \\ &= \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{1}{2\mathrm{i}\sin(2s\alpha)} \left(2\mathrm{i}\sin(s(\theta+\alpha))F_+(s) + 2\mathrm{i}\sin(s(\alpha-\theta))F_-(s)\right) r^{-s}\,\mathrm{d}s \\ &= \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\left(\sin(s(\alpha+\theta))F_+(s) + \sin(s(\alpha-\theta))F_-(s)\right) r^{-s}}{\sin(2s\alpha)}\,\mathrm{d}s. \end{aligned}$$

Example 4.4. Suppose ϕ satisfies

$$\nabla^2\phi = 0, \qquad 0 < r < \infty, \quad -\alpha < \theta < \alpha,$$

with

$$\phi(r,\pm\alpha) = f_\pm(r) = \begin{cases} 1 & \text{if } 0 < r < a, \\ 0 & \text{if } r > a. \end{cases}$$

Taking the Mellin transform gives

$$s^2\Phi(s,\theta) + \frac{\mathrm{d}^2\Phi}{\mathrm{d}\theta^2}(s,\theta) = 0,$$

so that

$$\Phi(s,\theta) = A(s)\mathrm{e}^{\mathrm{i}s\theta} + B(s)\mathrm{e}^{-\mathrm{i}s\theta}.$$

The Mellin transform of the boundary condition is

$$F_{\pm}(s) = \int_0^a r^{s-1}\,\mathrm{d}r = \left[\frac{r^s}{s}\right]_0^a = \frac{a^s}{s}, \qquad \mathrm{Re}(s) > 0.$$

The boundary conditions give

$$\begin{aligned} A(s)\mathrm{e}^{-\mathrm{i}s\alpha} + B(s)\mathrm{e}^{\mathrm{i}s\alpha} &= \frac{a^s}{s}, \\ A(s)\mathrm{e}^{\mathrm{i}s\alpha} + B(s)\mathrm{e}^{-\mathrm{i}s\alpha} &= \frac{a^s}{s}, \end{aligned}$$

so that A and B are given by

$$A(s) = B(s) = \frac{a^s}{s}\frac{\sin(s\alpha)}{\sin(2s\alpha)} = \frac{a^s}{2s\cos(s\alpha)}.$$

Thus

$$\Phi(s,\theta) = \frac{a^s\cos(s\theta)}{s\cos(s\alpha)},$$

which is holomorphic in the strip $0 < \mathrm{Re}(s) < \pi/(2\alpha)$. Thus

$$\phi(r,\theta) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{a^s\cos(s\theta)}{r^s s\cos(s\alpha)}\,\mathrm{d}s,$$

where $0 < c < \pi/(2\alpha)$. It is possible to show (see problem sheet 3) that

$$\phi = \begin{cases} 1 - \dfrac{2}{\pi}\tan^{-1}\left(\dfrac{2(r/a)^\beta\cos(\beta\theta)}{1-(r/a)^{2\beta}}\right) & \text{for } 0 < r < a, \\ \dfrac{2}{\pi}\tan^{-1}\left(\dfrac{2(r/a)^\beta\cos(\beta\theta)}{(r/a)^{2\beta}-1}\right) & \text{for } r > a, \end{cases}$$

where $\beta = \pi/(2\alpha)$.

■

5 Generation of transform pairs

5.1 Summary so far

$A\mathbf{x} = \mathbf{b}$	$I = \sum_{n=1}^{N} \mathbf{e}_n \mathbf{e}_n^T$	$A^{-1} = \sum_{n=1}^{N} \frac{\mathbf{e}_n \mathbf{e}_n^T}{\lambda_n}$
$Lu = f$	$\delta(x-\xi) = \sum_i \phi_i(\xi)\phi_i(x)$	$G(x,\xi) = \sum_{n=1}^{\infty} \frac{\phi_n(\xi)\phi_n(x)}{\lambda_n}$
$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \alpha^2 u = f$	$\delta(x-\xi) = \int_{-\infty}^{\infty} \phi_k(x)\phi_k^*(\xi)\,\mathrm{d}k$	$G(x,\xi) = \int_{-\infty}^{\infty} \frac{\phi_k(x)\phi_k^*(\xi)}{\lambda(k)}\,\mathrm{d}k$
$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} = f$	$\delta(r-\rho) = \sum_{n=1}^{\infty} r u_n(r) u_n(\rho)$	$G(r,\rho) = \sum_{n=1}^{\infty} \frac{r u_n(r) u_n(\rho)}{\lambda_n}$
$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} = f$	$\delta(r-\rho) = \int_0^{\infty} kr J_0(kr) J_0(k\rho)\,\mathrm{d}k$	$G(r,\rho) = \int_0^{\infty} \frac{kr J_0(kr) J_0(k\rho)}{\lambda(k)}\,\mathrm{d}k$
$r^2\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + r\frac{\mathrm{d}u}{\mathrm{d}r} = f$	$\delta(r-\rho) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} r^{-s}\rho^{s-1}\,\mathrm{d}s$	$G(r,\rho) = \frac{1}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{r^{-s}\rho^{s-1}}{\lambda(s)}\,\mathrm{d}s$

5.2 Generation of transform pairs

The transform pair arises from the resolution of the identity. For example, once we know

$$\delta(x-\xi) = \int_{-\infty} \phi_k(x)\phi_k^*(\xi)\,\mathrm{d}k$$

then if we define the forward transform as

$$\hat{u}(k) = \int_{-\infty}^{\infty} u(x)\phi_k(x)\,\mathrm{d}x$$

then the inverse is given by

$$u(\xi) = \int_{-\infty}^{\infty} \hat{u}(k)\phi_k^*(\xi)\,\mathrm{d}\xi$$

since

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{u}(k)\phi_k^*(\xi)\,\mathrm{d}k &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} u(x)\phi_k(x)\,\mathrm{d}x\right)\phi_k^*(\xi)\,\mathrm{d}k\\ &= \int_{-\infty}^{\infty} u(x)\left(\int_{-\infty}^{\infty}\phi_k(x)\phi_k^*(\xi)\,\mathrm{d}k\right)\mathrm{d}x\\ &= \int_{-\infty}^{\infty} u(x)\delta(x-\xi)\,\mathrm{d}x = u(\xi).\end{aligned}$$

So the question of the generation of transform pairs, becomes a question about the resolution of the identity. Now observe that the Green's function for

$$\frac{\mathrm{d}^2G}{\mathrm{d}x^2} - \alpha^2 G = \delta(x-\xi)$$

is

$$G(x,\xi) = \int_{-\infty}^{\infty} \frac{\phi_k(x)\phi_k^*(\xi)}{\lambda(k)}\,\mathrm{d}k.$$

Suppose we found instead the Green's function for the operator

$$\frac{\mathrm{d}^2G}{\mathrm{d}x^2} - \alpha^2 G - \mu G = \delta(x-\xi).$$

Then it is easy to see that this is

$$G(x,\xi;\mu) = \int_{-\infty}^{\infty} \frac{\phi_k(x)\phi_k^*(\xi)}{\lambda(k)-\mu}\,\mathrm{d}k.$$

Now, thinking of this as a function of complex μ, the integrand has a single simple pole at $\mu = \lambda(k)$. Thus if we integrate in a large circular contour in the complex plane C_∞ (which we will let tend to infinity) then

$$\int_{C_\infty} G(x,\xi;\mu)\,\mathrm{d}\mu = -2\pi\mathrm{i}\int_{-\infty}^{\infty}\phi_k(x)\phi_k^*(\xi)\,\mathrm{d}k = -2\pi\mathrm{i}\delta(x-\xi).$$

Thus if we can find $G(x,\xi;\mu)$ we can find a resolution of the identity. This works even when there is a continuous spectrum.

Example 5.1. Fourier transform Consider the operator

$$Lu = -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0,$$

in $-\infty < x < \infty$ with $u \in L^2$.

The Green's function we need to find satisfies

$$-\frac{\mathrm{d}^2 G}{\mathrm{d}x^2} - \mu G = \delta(x - \xi),$$

in $-\infty < x < \infty$ with $G \in L^2$. We calculate directly that, for μ not a positive real number,

$$G(x, \xi; \mu) = \begin{cases} A(\xi)\mathrm{e}^{-\mathrm{i}\sqrt{\mu}x}x & -\infty < x < \xi < \infty, \\ B(\xi)\mathrm{e}^{\mathrm{i}\sqrt{\mu}x} & -\infty < \xi < x < \infty, \end{cases}$$

where we choose the branch on which $\mathrm{Im}(\sqrt{\mu}) > 0$ in order for G to decay as $x \to \pm\infty$. The conditions at $x = \xi$ then give

$$\begin{aligned} A(x)\mathrm{e}^{-\mathrm{i}\sqrt{\mu}x} &= B(x)\mathrm{e}^{\mathrm{i}\sqrt{\mu}x}, \\ \mathrm{i}\sqrt{\mu}B(x)\mathrm{e}^{\mathrm{i}\sqrt{\mu}x} + \mathrm{i}\sqrt{\mu}A(x)\mathrm{e}^{-\mathrm{i}\sqrt{\mu}x} &= -1. \end{aligned}$$

Eliminating B gives

$$A(x) = -\frac{\mathrm{e}^{\mathrm{i}\sqrt{\mu}x}}{2\mathrm{i}\sqrt{\mu}}, \qquad B(x) = -\frac{\mathrm{e}^{-\mathrm{i}\sqrt{\mu}x}}{2\mathrm{i}\sqrt{\mu}},$$

so that

$$G(x, \xi; \mu) = \begin{cases} -\dfrac{\mathrm{e}^{\mathrm{i}\sqrt{\mu}\xi}\mathrm{e}^{-\mathrm{i}\sqrt{\mu}x}}{2\mathrm{i}\sqrt{\mu}} & -\infty < x < \xi < \infty, \\ -\dfrac{\mathrm{e}^{-\mathrm{i}\sqrt{\mu}\xi}\mathrm{e}^{\mathrm{i}\sqrt{\mu}x}}{2\mathrm{i}\sqrt{\mu}} & -\infty < \xi < x < \infty, \end{cases}$$

Now, we have to integrate G around the large circle C_∞. However, there is a branch point at the origin and a branch cut along the positive real axis. The origin is the only singularity of G though, so that we can deform our contour to a contour C_2 which lies on the branch cut, that is, it goes from ∞ to 0 along the top of the positive real axis and then back out to ∞ along the bottom of the positive real axis. We make the change of variables $\mu = \zeta^2$ so that this contour is unfolded and the path of integration is simply $\zeta = \infty$ to $\zeta = -\infty$.

Thus, for $x < \xi$,

$$\begin{aligned}\delta(x-\xi) &= -\frac{1}{2\pi \mathrm{i}}\int_{C_\infty} G(x,\xi;\mu)\,\mathrm{d}\mu \\ &= -\frac{1}{2\pi}\int_{C_2} \frac{\mathrm{e}^{\mathrm{i}\sqrt{\mu}\xi}\mathrm{e}^{-\mathrm{i}\sqrt{\mu}x}}{2\sqrt{\mu}}\,\mathrm{d}\mu \\ &= -\frac{1}{2\pi}\int_{\infty}^{-\infty} \frac{1}{2\zeta}\mathrm{e}^{\mathrm{i}\zeta\xi}\mathrm{e}^{-\mathrm{i}\zeta x}\,2\zeta\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\zeta\xi}\mathrm{e}^{-\mathrm{i}\zeta x}\,\mathrm{d}\zeta.\end{aligned}$$

This representation of the delta function gives the Fourier transform pair:

$$F(\zeta) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}\zeta x}\,\mathrm{d}x, \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\zeta)\mathrm{e}^{\mathrm{i}\zeta\xi}\,d\zeta.$$

■

Example 5.2. Fourier sine series

Consider the operator

$$Lu = -\frac{\mathrm{d}^2u}{\mathrm{d}x^2} = 0$$

with $u(0) = u(1) = 0$. The Green's function for $(L-\mu)u$ is (see example 1.1)

$$G(x,\xi;\mu) = \begin{cases} \dfrac{\sin\sqrt{\mu}x\,\sin\sqrt{\mu}(1-\xi)}{\sqrt{\mu}\sin\sqrt{\mu}} & \text{if } 0 \le x < \xi \le 1, \\ \dfrac{\sin\sqrt{\mu}\xi\,\sin\sqrt{\mu}(1-x)}{\sqrt{\mu}\sin\sqrt{\mu}} & \text{if } 0 \le \xi < x \le 1. \end{cases}$$

There are simple poles at $\mu = n^2\pi^2$ for $n = 1, 2, \ldots$. Note that $\mu = 0$ is not a pole or branch point since for small μ

$$G(x,\xi;\mu) = \begin{cases} x(1-\xi) & \text{if } 0 \le x < \xi \le 1, \\ \xi(1-x) & \text{if } 0 \le \xi < x \le 1. \end{cases}$$

Applying the residue theorem we find that (for $x < \xi$)

$$\begin{aligned}\delta(x-\xi) &= -\frac{1}{2\pi \mathrm{i}}\int_{C_\infty} G(x,\xi;\mu)\,\mathrm{d}\mu \\ &= -2\sum_{k=1}^{\infty} \left.\frac{\sin\sqrt{\mu}x\,\sin\sqrt{\mu}(1-\xi)}{\cos\sqrt{\mu}}\right|_{\mu=n^2\pi^2} \\ &= 2\sum_{k=1}^{\infty} \sin k\pi x\,\sin k\pi\xi.\end{aligned}$$

This representation of the delta function is equivalent to the Fourier sine series, since

$$\begin{aligned} f(x) &= \int_0^1 \delta(x-\xi) f(\xi)\,\mathrm{d}\xi \\ &= \sum_{k=1}^{\infty} \sin k\pi x \left(2 \int_0^1 f(\xi) \sin k\pi\xi \,\mathrm{d}\xi \right) \\ &= \sum_{k=1}^{\infty} \alpha_k \sin k\pi x, \end{aligned}$$

where

$$\alpha_k = 2 \int_0^1 f(\xi) \sin k\pi\xi \,\mathrm{d}\xi$$

is the usual Fourier coefficient.

■

Example 5.3. Fourier sine integral transform

Consider the operator

$$Lu = -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} = 0,$$

in $0 \le x < \infty$ with $u(0) = 0$ with $u \in L^2[0, \infty)$.

The Green's function we need to find satisfies

$$-\frac{\mathrm{d}^2 G}{\mathrm{d}x^2} - \mu G = \delta(x - \xi),$$

in $0 \le x < \infty$ with $G(0) = 0$ with $G \in L^2[0, \infty)$. We calculate directly that, for μ not a positive real number,

$$G(x, \xi; \mu) = \begin{cases} A(\xi) \sin \sqrt{\mu} x & 0 \le x < \xi < \infty, \\ B(\xi) \mathrm{e}^{\mathrm{i}\sqrt{\mu} x} & 0 \le \xi < x < \infty, \end{cases}$$

where we have imposed the conditions at $x = 0$ and $x \to \infty$. We choose the branch on which $\mathrm{Im}(\sqrt{\mu}) > 0$ in order for G to decay as $x \to \infty$. The conditions at $x = \xi$ then give

$$\begin{aligned} A(x) \sin \sqrt{\mu} x &= B(x) \mathrm{e}^{\mathrm{i}\sqrt{\mu} x}, \\ \mathrm{i}\sqrt{\mu} B(x) \mathrm{e}^{\mathrm{i}\sqrt{\mu} x} - A(x) \sqrt{\mu} \cos \sqrt{\mu} x &= -1. \end{aligned}$$

Eliminating B gives

$$A(x) = -\frac{1}{\mathrm{i}\sqrt{\mu}\sin\sqrt{\mu}x - \sqrt{\mu}\cos\sqrt{\mu}x} = \frac{1}{\sqrt{\mu}\left(\cos\sqrt{\mu}x - \mathrm{i}\sin\sqrt{\mu}x\right)} = \frac{\mathrm{e}^{\mathrm{i}\sqrt{\mu}x}}{\sqrt{\mu}}.$$

Thus

$$B(x) = \frac{\sin\sqrt{\mu}x}{\sqrt{\mu}},$$

and therefore

$$G(x,\xi;\mu) = \begin{cases} \dfrac{1}{\sqrt{\mu}}\sin\sqrt{\mu}x\,\mathrm{e}^{\mathrm{i}\sqrt{\mu}\xi} & 0 \le x < \xi < \infty, \\ \dfrac{1}{\sqrt{\mu}}\sin\sqrt{\mu}\xi\,\mathrm{e}^{\mathrm{i}\sqrt{\mu}x} & 0 \le \xi < x < \infty. \end{cases}$$

Now, we have to integrate G around the large circle C_∞. However, there is a branch point at the origin and a branch cut along the positive real axis. The origin is the only singularity of G though, so that we can deform our contour to a contour C_2 which lies on the branch cut, that is, it goes from ∞ to 0 along the top of the positive real axis and then back out to ∞ along the bottom of the positive real axis. We make the change of variables $\mu = \zeta^2$ so that this contour is unfolded and the path of integration is simply $\zeta = \infty$ to $\zeta = -\infty$.

Note also that

$$\frac{1}{\sqrt{\mu}}\sin\sqrt{\mu}x \qquad \text{and} \qquad \cos\sqrt{\mu}x$$

are entire functions of μ and therefore integrate to zero around any closed contour. Thus

$$\frac{1}{\sqrt{\mu}}\sin\sqrt{\mu}x\cos\sqrt{\mu}\xi \qquad \text{and} \qquad \frac{1}{\sqrt{\mu}}\sin\sqrt{\mu}\xi\cos\sqrt{\mu}x$$

integrate to zero and we need only consider the integral of

$$\frac{\mathrm{i}}{\sqrt{\mu}}\sin\sqrt{\mu}x\sin\sqrt{\mu}\xi.$$

Thus

$$\begin{aligned}
\delta(x-\xi) &= -\frac{1}{2\pi \mathrm{i}} \int_{C_\infty} G(x,\xi;\mu)\,\mathrm{d}\mu \\
&= -\frac{1}{2\pi} \int_{C_2} \frac{1}{\sqrt{\mu}} \sin\sqrt{\mu}x \sin\sqrt{\mu}\xi\,\mathrm{d}\mu \\
&= -\frac{1}{2\pi} \int_{\infty}^{-\infty} \frac{1}{\zeta} \sin(\zeta x)\sin(\zeta\xi)\,2\zeta\,\mathrm{d}\zeta \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} \sin(\zeta x)\sin(\zeta\xi)\,\mathrm{d}\zeta \\
&= \frac{2}{\pi} \int_{0}^{\infty} \sin(\zeta x)\sin(\zeta\xi)\,\mathrm{d}\zeta
\end{aligned}$$

This representation of the delta function gives the Fourier sine integral transform:

$$F(\zeta) = \int_0^\infty f(x)\sin(\zeta x)\,\mathrm{d}x, \qquad f(x) = \frac{2}{\pi}\int_0^\infty F(\zeta)\sin(\zeta x)\,d\zeta.$$

■

Different operators may give the same transform pair. For example, any constant coefficient linear differential operator in $(-\infty,\infty)$ will give the Fourier transform pair. This is why the Fourier transform is a good solution method for all such equations. We used the operator

$$Lu = \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - \alpha^2 u = 0$$

to derive the Fourier transform, but often the operator

$$Lu = -\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}$$

is used.