MMSC Further PDEs HT2016 — Sheet 1

1. Suppose that the real, symmetric $N \times N$ matrix A has precisely one zero eigenvalue λ_1, with eigenvector $\mathbf{e}_1$. Show that when $\mathbf{e}_1^T\mathbf{b} \neq 0$ there is no solution to $A\mathbf{x} = \mathbf{b}$, while when $\mathbf{e}_1^T\mathbf{b} = 0$ the solution is

$$\mathbf{x} = \sum_{n=2}^{N} \frac{\mathbf{e}_n\mathbf{e}_n^T}{\lambda_n}\mathbf{b} + \alpha\mathbf{e}_1,$$

where α is an arbitrary constant.

2. Show that the Green's function $G(x;\xi)$ for the forced one-dimensional Helmholtz equation satisfying

$$\frac{\mathrm{d}^2G}{\mathrm{d}x^2} + k^2G = \delta(x-\xi), \quad k \notin \mathbb{Z}, \quad G(0;\xi) = G(\pi;\xi) = 0,$$

is

$$G(x;\xi) = \begin{cases} -\dfrac{\sin(kx)\sin(k(\pi-\xi))}{k\sin k\pi} & 0 \leq x < \xi \leq \pi, \\ -\dfrac{\sin(k\xi)\sin(k(\pi-x))}{k\sin k\pi} & 0 \leq \xi < x \leq \pi. \end{cases}$$

3. Show that an eigenfunction expansion gives the solution of

$$\frac{\mathrm{d}^2U}{\mathrm{d}x^2} + \frac{U}{4} = \frac{x}{2}, \qquad U(0) = U(\pi) = 0,$$

as

$$U(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(1/4-n^2)}\sin nx.$$

Show that this is the Fourier sine series of the solution obtained by conventional means.

4. Show that any "well-behaved" function $f(r)$ in the interval $0 < r < 1$ may be expanded as the series

$$f(r) = \sum_{n=1}^{\infty} c_n J_\nu(\alpha_n r)$$

where the constants $\alpha_1 < \alpha_2 < \alpha_3 \cdots$ are given by the roots $J_\nu(\alpha_n) = 0$, and the coefficients c_n are given by

$$c_n = \frac{2}{J_{\nu+1}(\alpha_n)^2} \int_0^1 r f(r) J_\nu(\alpha_n r)\, \mathrm{d}r.$$

[*Hint: consider the eigenfunction problem*

$$\phi''(r) + \frac{1}{r}\phi'(r) - \frac{\nu^2}{r^2}\phi(r) = -\alpha^2\phi(r), \qquad \phi(0) = \phi(1) = 0.$$

Completeness of the eigenfunctions can be assumed. For the normalisation condition consider the usual Sturm-Liouville argument for the integral $\int_0^1 r(\alpha^2 - \alpha_n^2)\phi\phi_n\, \mathrm{d}r$ *for* α *not an eigenvalue and then let* $\alpha \to \alpha_n$.

The Bessel function $J_\nu(r)$ *satisfies the differential equation*

$$J_\nu''(r) + \frac{1}{r}J_\nu'(r) + \left(1 - \frac{\nu^2}{r^2}\right) J_\nu(r) = 0,$$

with $J_\nu(0) = 0$ *for* $\nu > 0$, *and the identities*

$$2J_\nu'(x) = J_{\nu-1}(x) - J_{\nu+1}(x), \qquad \frac{2\nu}{x}J_\nu(x) = J_{\nu-1}(x) + J_{\nu+1}(x).]$$

5. Investigate the behaviour of the partial sums of the representation of the identity

$$\sum_{n=1}^{N} \phi_n(x)\phi_n(\xi)$$

for increasing numbers of terms N, and your favourite set of orthonormal functions ϕ_n, using suitable mathematical software.

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