MMSC Further PDEs HT2016 — Sheet 2

1. Suppose $f(z)$ is a function whose Fourier transform $F(k)$ is zero for $|k| > \pi/h$. The function $f(z)$ is said to be **band limited** since its component frequencies are limited to the band $|k| \leq \pi/h$. On the interval $-\pi/h < k < \pi/h$ we can represent $F(k)$ as a Fourier series

$$F(k) = \sum_{n=-\infty}^{\infty} a_n \mathrm{e}^{\mathrm{i}nhk}, \qquad a_n = \frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} F(k)\,\mathrm{e}^{-\mathrm{i}nhk}\,\mathrm{d}k.$$

(i) By taking an inverse Fourier transform show that

$$f(t) = \frac{1}{h}\sum_{n=-\infty}^{\infty} a_n \operatorname{sinc}\left(\frac{t}{h} - n\right),$$

where

$$\operatorname{sinc}(t) = \frac{\sin \pi t}{\pi t}.$$

(ii) Deduce that

$$f(t) = \sum_{n=-\infty}^{\infty} f(nh) \operatorname{sinc}\left(\frac{t}{h} - n\right).$$

[This means that a band limited function can be represented **exactly** by the sum above if we know its values at the equally spaced mesh points $t = nh$. Since the human auditory system is band limited (we can hear frequencies between about 2 and 20kHz) this means that audible signals can be reconstructed very accurately from knowledge of the signal at equally spaced times. This is the basis of digital sound recording.]

2. Find the Fourier transform of the Bessel function

$$J_0(ar) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{\mathrm{i}ar\cos\theta}\,\mathrm{d}\theta,$$

and show that it is band limited.

3. Solve the integral equation

$$4\int_{-\infty}^{\infty} \mathrm{e}^{-|x-y|}u(y)\,\mathrm{d}y + u(x) = f(x), \qquad -\infty < x < \infty,$$

for u given f.

4. (i) If $\tilde{f}(p) = \mathcal{L}[f; p]$ denotes the Laplace transform of $f(x)$, and

$$h(x) = \int_0^x g(x-y) f(y) \, \mathrm{d}y,$$

show that

$$\tilde{h}(p) = \tilde{g}(p) \tilde{f}(p).$$

(ii) Show that

$$\mathcal{L}[\sin ax; p] = \frac{a}{a^2 + p^2}.$$

(iii) Use the Laplace transform to solve the Volterra integral equation

$$3a \int_0^x \sin(a(x-y)) \, u(y) \, \mathrm{d}y + u(x) = f(x), \qquad 0 \le x < \infty,$$

for u given f.

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