MMSC Further PDEs HT2016 — Sheet 3

1. Let ϕ satisfy Laplace's equation in the half-space $z \geq 0$. Suppose that $\phi(r, 0) = f(r)$ on the boundary $z = 0$ and that $\phi \to 0$ as $z \to \infty$. Show that

$$\phi(r, z) = \int_0^\infty \hat{f}(k) J_0(kr) \mathrm{e}^{-kz} k \, \mathrm{d}k$$

where $\hat{f}$ is the Hankel transform of f.

For the special case in which

$$f(r) = \begin{cases} 1 & \text{for } r \leq a, \\ 0 & \text{for } r > a, \end{cases}$$

show that

$$\phi(r, z) = a \int_0^\infty J_1(ka) J_0(kr) \mathrm{e}^{-kz} \, \mathrm{d}k.$$

[*Note that Bessel functions satisfy the recurrence relations*

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x), \qquad \frac{2n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x).]$$

2. Consider a point charge a distance a from a single earthed plate. The potential ϕ satisfies

$$\nabla^2 \phi = -4\pi \delta(x) \delta(y) \delta(z)$$

with $\phi = 0$ on $z = -a$ and $\phi \to 0$ as $z \to \infty$. Show that

$$\phi = 2 \int_0^\infty \mathrm{e}^{-ka} \sinh k(z + a) J_0(kr) \, \mathrm{d}k$$

if $-a < z < 0$, and find the corresponding expression for $z > 0$.

3. Let $F(s) = \mathcal{M}[f(x); s]$ be the Mellin transform of f.

(i) Show that, for $a \neq 0$ real

$$\mathcal{M}[f(x^a); s] = \frac{1}{|a|} F\left(\frac{s}{a}\right), \qquad \mathcal{M}[f(ax)] = \frac{F(s)}{a^s}, \qquad \mathcal{M}[(\log x) f(x)] = \frac{\mathrm{d}}{\mathrm{d}s} F(s).$$

(ii) Show that

$$\mathcal{M}\left[\int_x^\infty f(t)\,\mathrm{d}t; s\right] = \frac{F(s+1)}{s},$$

provided $s^{-1} x^s \int_x^\infty f(t)\,\mathrm{d}t = 0$ when $x = 0$ and $x = \infty$.

(iii) Let $f(r)$ be real valued and suppose f can be analytically continued into a function $f(z)$, $z = r\mathrm{e}^{\mathrm{i}\theta}$ in some sector $|\theta| < \beta$ of the complex plane. Show that

$$\mathcal{M}\left[f(r\mathrm{e}^{\mathrm{i}\theta}); s\right] = \mathrm{e}^{-\mathrm{i}\theta s} \mathcal{M}[f(r); s]$$

where the Mellin transforms are with respect to r. Deduce that

$$\mathcal{M}\left[\mathrm{Re}\left(f(r\mathrm{e}^{\mathrm{i}\theta})\right); s\right] = \cos(s\theta) \mathcal{M}[f(r); s]$$

$$\mathcal{M}\left[\mathrm{Im}\left(f(r\mathrm{e}^{\mathrm{i}\theta})\right); s\right] = -\sin(s\theta) \mathcal{M}[f(r); s]$$

4. Show that

$$\mathcal{M}[(1+x)^{-1}; s] = \frac{\pi}{\sin \pi s}.$$

[*Hint: put the branch cut of x^{s-1} along the real axis and consider the integral from infinity to zero on the underside of the real axis, clockwise around the branch point at zero, and out to infinity along the upper side of the real axis.*]

5. Using the properties of the Mellin transform in Q3, starting from the transform in Q4, show that the inverse Mellin transform of

$$\Phi(s,\theta) = \frac{a^s \cos(s\theta)}{s\cos(s\alpha)}$$

is

$$\phi(r,\theta) = \begin{cases} 1 - \dfrac{1}{\pi}\tan^{-1}\left(\dfrac{2(r/a)^\beta \cos(\beta\theta)}{1-(r/a)^{2\beta}}\right) & \text{for } 0 < r < a, \\ \dfrac{1}{\pi}\tan^{-1}\left(\dfrac{2(r/a)^\beta \cos(\beta\theta)}{(r/a)^{2\beta}-1}\right) & \text{for } r > a, \end{cases}$$

where $\beta = \pi/(2\alpha)$.

[*You may need the facts that*

$$\int_x^\infty \frac{\mathrm{d}t}{1+t^2} = \frac{\pi}{2} - \tan^{-1} x,$$

$$\tan^{-1} x = \frac{1}{2\mathrm{i}} \log\left(\frac{1+\mathrm{i}x}{1-\mathrm{i}x}\right).]$$

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