

MMSC Further PDEs HT2016 — Sheet 4

1. Consider the sum

$$G(x) = \sum_{k=1}^{\infty} k e^{-k^{1/2}x}.$$

Use the Mellin transform to show that

$$G(x) = \frac{12}{x^4} + \sum_{m=0}^M \frac{(-1)^m}{m!} \zeta\left(-\frac{m}{2} - 1\right) x^m + o(x^M)$$

as  $x \rightarrow 0$  for any  $M \geq 0$ , where  $\zeta(s)$  is the Riemann zeta function.

2. Use the Green's function for  $(L - \mu)u$  to find the spectral representation of the delta function for the following operators

- (i)  $Lu = -u''$ ,  $x \in [0, 1]$ ,  $u(0) = u'(1) = 0$ ;
- (ii)  $Lu = -u''$ ,  $x \in [0, \infty)$ ,  $u'(0) = 0$ ;
- (iii)  $Lu = -u''$ ,  $x \in [0, \infty)$ ,  $u'(0) = \alpha u(0)$ .

Deduce the transform pair in each case.