Complex Analysis Revision Notes

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1 Differentiable, holomorphic

It all begins with the innocuous definition that a function $f(z)$ of the complex variable z is *differentiable* at the point z if

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} \tag{1}$$

exists however $h \to 0$; and $f(z)$ is *holomorphic* (some people use *analytic*) in a region D if it is differentiable at each point of D.

Writing $z = x + \mathrm{i}y$, $f = u + \mathrm{i}v$ where x, y, u and v are real, and taking h real gives

$$f'(z) = \lim_{h\to 0}\left(\frac{u(x+h,y)-u(x,y)}{h} + \mathrm{i}\frac{v(x+h,y)-v(x,y)}{h}\right) = \frac{\partial u}{\partial x} + \mathrm{i}\frac{\partial v}{\partial x}.$$

Now taking h imaginary by writing $h = \mathrm{i}\delta$ with δ real gives

$$f'(z) = \lim_{\delta\to 0}\left(\frac{u(x,y+\delta)-u(x,y)}{\mathrm{i}\delta} + \mathrm{i}\frac{v(x,y+\delta)-v(x,y)}{\mathrm{i}\delta}\right) = \frac{1}{\mathrm{i}}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Setting these two values of $f'(z)$ equal, we find the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

relating the real and imaginary parts of $f(z)$ wherever $f(z)$ is differentiable.

Cross-differentiating the Cauchy–Riemann equations shows that u and v are solutions of Laplace's equation (i.e. they are harmonic functions):

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

This is enormously important in applications, because Laplace's equation arises very frequently, and we can use complex functions to solve boundary value problems for it (in two dimensions, at least).

2 Integrals

The integral of a function of z along a curve Γ, which may be open or closed, is defined parametrically in the obvious way. If the path Γ is given parametrically by $z = z(t)$, $t_0 < t < t_1$, then

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_{t_0}^{t_1} f(z(t))\frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t.$$

3 Cauchy's theorem and path independence

Having defined integrals of a function of z along a curve by parametrising the curve, we can state the mainspring of complex analysis, Cauchy's theorem:

If a function $f(z)$ is holomorphic within, and continuous on, on a simple curve Γ, then

$$\int_\Gamma f(z)\, dz = 0.$$

It is an immediate consequence of Cauchy's theorem that if Γ_1 and Γ_2 are two curves joining the point z_0 to another point z_1, and if $f(z)$ is holomorphic in a region containing Γ_1, Γ_2 and the region between them, then

$$\int_{\Gamma_1} f(z)\,\mathrm{d}z = \int_{\Gamma_2} f(z)\,\mathrm{d}z,$$

so that the integral is path-independent. This is often stated as the **deformation theorem**: if one contour Γ_1 can be deformed smoothly into another one Γ_2 while crossing only points at which $f(z)$ is holomorphic, then the integral of $f(z)$ along Γ_1 is equal to the integral along Γ_2.

4 Cauchy's integral formula

Take a simple closed contour Γ, and let $f(z)$ be holomorphic on Γ and inside it. Then then values of $f(z)$ on Γ determine its values at all points within Γ as well, via **Cauchy's integral formula**: for all z within Γ,

$$f(z) = \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{f(t)}{t-z}\,\mathrm{d}t.$$

The proof is simple, by deforming the contour to a small circle surrounding z and adding and subtracting $f(z)$:

$$\oint_\Gamma \frac{f(t)}{t-z}\,\mathrm{d}t. = \oint_{|t-z|=\epsilon} \frac{f(t)}{t-z}\,\mathrm{d}t = \oint_{|t-z|=\epsilon} \frac{f(z)}{t-z}\,\mathrm{d}t + \oint_{|t-z|=\epsilon} \frac{f(t)-f(z)}{t-z}\,\mathrm{d}t;$$

the first integral on the right is equal to $2\pi\mathrm{i}f(z)$ and the second vanishes as $\epsilon \to 0$ by continuity of f.

5 Infinite differentiability!

Given that

$$f(z) = \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{f(t)}{t-z}\,\mathrm{d}t,$$

it is tempting to differentiate with respect to z under the integral sign to find

$$f'(z) = \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{f(t)}{(t-z)^2}\,\mathrm{d}t$$

and the justification of this, via $(f(z+h)-f(z))/h$, is not difficult. But then, we can differentiate again (with essentially the same justification), to find that $f''(z)$ exists and is equal to

$$\frac{2}{2\pi \mathrm{i}} \oint_\Gamma \frac{f(t)}{(t-z)^3}\,\mathrm{d}t.$$

and we have effortlessly established that once a complex function of z is differentiable, so is its derivative! Hence, holomorphic functions are infinitely differentiable. The contrast with real analysis is very marked. Indeed, all the interest in complex analysis is focused on the points where functions fail to be holomorphic, known as singularities or singular points.

Furthermore, we have a formula for the derivatives: continuing to differentiate under the integral sign, we have

$$f^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \oint_\Gamma \frac{f(t)}{(t-z)^{n+1}}\,\mathrm{d}t,$$

this is rarely used *per se*, but it is the key to Taylor's theorem.

6 Liouville

A function is called *entire* if it is holomorphic in the whole complex plane (eg z, e^z). Such a function must have a singularity at infinity, because of:

Liouville's theorem. *Any bounded entire function $f(z)$ is constant.*

That is, if $|f(z)| < M$ for some M and all z, then f is a constant (less than M in modulus). The proof is by looking at Cauchy's integral formula for $f'(z)$ and taking Γ to be a large circle; letting the radius of the circle tend to infinity, we have $f'(z) = 0$.

7 Taylor

Knowing that a holomorphic function has derivatives of all orders, we expect it to have power series representation. It does:

Taylor's theorem. *If $f(z)$ is holomorphic in a disc $D(a;R)$, then there is a series representation*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

which converges to $f(z)$ for all $0 \le |z-a| < R$. Moreover,

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Note that the series converges and it converges to $f(z)$; the latter need not be true in real analysis (eg the function e^{-1/x^2} has the Taylor series 0 at the origin, as all its derivatives exist but vanish there).

The circle of convergence for the series is the largest disc within which the series converges, and so the radius of convergence is the distance from $z = a$ to the nearest singular point of $f(z)$. The series diverges for $|z-a| > R$, while on the circle of convergence it may converge at some points (but must diverge at at least one).

8 Laurent

Taylor's theorem gives a series in ascending powers of z for a function holomorphic in a disc $|z-a| < R$. If, by contrast, we have a function $f(z)$ which is holomorphic for $S < |z-a| < \infty$ and, for definiteness, vanishes at infinity, we can form a series in *descending* powers of z, by finding the Taylor series of $g(z) = f(1/z)$ (thereby finding expressions for the coefficients as limits at infinity of powers of z times derivatives of $f(z)$). Alternatively by adapting the proof of Taylor's theorem so that Γ is a circle $S < |t| < |z|$ (again, $a = 0$ wlog) and expanding in powers of t/z, we have representations for the coefficients as integrals round Γ.

More generally, if our function is holomorphic in an annulus, we have:

Laurent's theorem. *If $f(z)$ is holomorphic in the annulus $S < |z-a| < R$, then in that annulus it has a series representation*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n,$$

where

$$c_n = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{f(t)}{(t-a)^{n+1}}\, \mathrm{d}t;$$

for $n \geq 0$, Γ is a circle $|z-a| < |t-a| < R$, while for $n < 0$, Γ is a circle $S < |t-a| < |z-a|$.

The part of the sum containing the negative powers is called the *principal part* of $f(z)$ at $z = a$, and it is holomorphic for $S < |z-a| < \infty$. The part containing positive powers is holomorphic for $0 \leq |z-a| < R$.

9 Classification of singularities

Suppose that $S = 0$ in Laurent's theorem, so that $f(z)$ is holomorphic for $0 < |z-a| < R$ (it may happen that $R = \infty$). Then $f(z)$ has an *isolated singularity* at $z = a$. These singularities can be classified into three categories, as follows.

1. If all the negative coefficients in the Laurent expansion vanish, then $f(z)$ can be made holomorphic at $z = a$ by setting $f(a) = c_0 = \lim_{z \to a} f(z)$. Such a singularity is termed *removable.*

2. If there is an integer $m > 0$ such that $c_{-m} \neq 0$ but $c_n = 0$ for $n < -m$, then $f(z)$ has a *pole of order* m at $z = a$. In this case the leading order behaviour of $f(z)$ near a is $c_{-m}(z-a)^{-m}$, and $(z-a)^m f(z)$ is holomorphic at $z = a$. A function whose only singularities are poles is called *meromorphic.*

3. If neither of the above holds, then there are infinitely many nonzero negative Laurent coefficients: the principal part goes on for ever. In this case $f(z)$ has an *isolated essential singularity* at $z = a$.

The behaviour of $f(z)$ at infinity is classified according to the behaviour of $f(1/z)$ near $z = 0$; thus, for example, z has a pole of order 1 at infinity, while e^z has an essential singularity there.

10 The residue theorem

We now turn Laurent's theorem round: instead of evaluating Laurent coefficients in terms of integrals, if $f(z)$ has an isolated singularity at $z = a$ and Γ encloses a, then

$$\begin{aligned}\oint_\Gamma f(z)\,\mathrm{d}z &= \oint_\Gamma \sum_{n=-\infty}^{\infty} c_n (z-a)^n\,\mathrm{d}z \\ &= \sum_{n=-\infty}^{\infty} c_n \oint_\Gamma (z-a)^n\,\mathrm{d}z \\ &= 2\pi \mathrm{i} c_{-1},\end{aligned}$$

as all the other integrals vanish (the powers $(t-a)^{-(n+1)}$ integrate to other powers, while $(t-a)^{-1}$ gives a log). The constant c_{-1} is the *residue* of $f(z)$ at $z = a$, so-called (presumably) because it is all that is left after integration.

The result is easily generalised to the case when $f(z)$ has several isolated singularities inside Γ, and is known as:

The Residue theorem. *If $f(z)$ is holomorphic inside Γ with the exception of a finite number of isolated singularities at $z = a_j$, then*

$$\oint_\Gamma f(z)\,\mathrm{d}z = \sum_j \mathrm{res}_{a_j} f(z).$$

Calculation of residues relies on a few variations on the theme of calculating local expansions. Apart from functions such as $\mathrm{e}^{1/z}$ for which we just calculate the power series, we may often have functions with poles, in the form

$$f(z) = \frac{g(z)}{h(z)}$$

where both $g(z)$ and $h(z)$ are holomorphic at $z = a$ and $h(a) = 0$, $g(a) \neq 0$. The order of the pole then depends on the order of the zero of $h(z)$ at $z = a$. If $h(z) = z - a$ the pole is a simple one and the residue is $g(a)$, and if $h(z) = (z-a)^n$, the Taylor expansion of $g(z)$ shows that the residue is $g^{(n-1)}(a)/(n-1)!$. If $h(a) = 0$ but $h'(a) \neq 0$, expanding $h(z) = (z-a)h'(a) + \cdots$ shows that the residue is $g(a)/h'(a)$; and so on.

11 Evaluation of integrals

Sometimes the original integral can be transformed into an integral round a closed contour, which may be evaluated using the residue theorem, but in other cases the contour must be made into a closed one by addition of a suitable return path; the integral along this must be estimated and shown to vanish in a suitable limit.

12 Fourier and Laplace transforms

The Fourier transform of a real function $f(x)$ is

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{ikx} \, dx,$$

and the inverse is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k) e^{-ikx} \, dk$$

(note the asymmetric position of the factor $1/2\pi$; not all authors put it here, so watch out for variations). Inversion is usually accomplished by contour integration in the k-plane.

Integration by parts shows that

$$\widehat{\frac{df}{dx}}(k) = -i\hat{f}(k),$$

and differentiation under the integral sign leads to

$$\widehat{(xf)}(k) = -i\frac{d\hat{f}}{dk}.$$

The Laplace transform operates on functions defined on the positive real axis:

$$\tilde{f}(p) = \int_{0}^{\infty} f(x) e^{-px} \, dx,$$

and if $f(x)e^{-\gamma x}$ is integrable (so that $|f(x)|$ grows no worse than $e^{\gamma x}$ as $x \to \infty$), then $\tilde{f}(p)$ exists for $\mathrm{Re}\, p \geq \gamma$ and is holomorphic in p for $\mathrm{Re}\, p > \gamma$; it can usually (being given by a formula) be analytically continued into the rest of the complex p-plane, although singularities inevitably occur. The inversion formula is

$$f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \tilde{f}(p) e^{px} \, dp.$$

The contour is usually (but not always) completed in the left-hand half-plane and in many problems the solution is given by a sum of residues from the interior of the completed contour, although sometimes a branch cut is also present in $\tilde{f}(p)$ and the solution is reduced to an integral along this cut.

13 Multiple valued functions (multifunctions)

A function $f(z)$ has a branch point at $z = a$ if, on taking a circuit round a, the final value of $f(z)$ is not equal to the original one. Examples are $f(z) = z^{\frac{1}{2}}$, where a circuit round the origin takes us from one branch of the square root to the other (from the ‘plus’ root to the ‘minus’ root or vice versa), or $f(z) = \log z$, for which an anticlockwise circuit around the origin leaves the real part unchanged but increases the imaginary part by 2π (its multivaluedness stems from the ambiguity in the definition of $\arg z$). These functions also both have branch points at infinity (infinity is a branch point for $f(z)$ if the origin is a branch point for $f(1/z)$).

There are two solutions to this difficulty. One is to extend the domain of definition of the function by constructing its Riemann surface, on which the function is single-valued and holomorphic everywhere except at the branch points (and any other singularities). For example, the Riemann surface for $z^{\frac{1}{2}}$ consists of two copies of the complex plane ('sheets') joined together at the origin and at infinity, and passing through each other in such a way that a complete circuit of the origin takes us from one sheet to the other. The Riemann surface for $\log z$ is like a multistory carpark.

The second solution is to restrict the domain of definition of the function so that the problematic circuits are forbidden. This is achieved by introducing *branch cuts*, joining the branch points, across which contours may not pass. Then it is possible to define single-valued *branches* of the (multi)function, which is regarded as the collection of these branches. For example, we can make $z^{\frac{1}{2}}$ single-valued by putting a cut along the negative real axis, and defining the two branches to be $r^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\theta/2}$ and $-r^{\frac{1}{2}}\mathrm{e}^{\mathrm{i}\theta/2}$, where $r = |z|$ and $\theta = \arg z$ is restricted so that $-\pi < \theta \leq \pi$. There is no need to take the cut along the negative real axis; any curve joining 0 to ∞ will do, and the choice is problem-dependent. With the cut again along the negative real axis and the same restriction on θ, the set of branches of $\log z$ is $\{\log r + \mathrm{i}\theta + 2k\pi\mathrm{i}\}$, $k \in \mathbb{Z}$; the branch with this cut and $k = 0$ is sometimes called the *principal branch*, written $\operatorname{Log} z$; the corresponding branch of $\arg z$, which is θ above, is written $\operatorname{Arg} z$.