GEOMETRY – SHEET 6 – Parametrized Surfaces

(Exercises on lectures in Week 7)

1. Planar parabolic co-ordinates u, v are defined by

$$x = \frac{u^2 - v^2}{2}, \qquad y = uv.$$

(i) Show that $x + iy = (u + iv)^2/2$. Deduce that as u, v vary over the positive numbers then (x, y) varies over the upper half-plane $y > 0$.

(ii) Sketch the curves $u =$ const. and $v =$ const. Show that these curves intersect at right angles.

2. Consider the hyperboloid of one sheet with equation $x^2 + y^2 = z^2 + 1$.

(i) Show that this hyperboloid can be parametrized as

$$\mathbf{r}(\theta, \lambda) = (\cos\theta, \sin\theta, 0) + \lambda\,(\sin\theta, -\cos\theta, 1) \qquad 0 \leqslant \theta < 2\pi,\ \lambda \in \mathbb{R}.$$

(ii) Determine a normal vector at $\mathbf{r}(\theta, \lambda)$ by working out $\mathbf{r}_\theta \wedge \mathbf{r}_\lambda$.

(iii) Determine a normal vector at $\mathbf{r}(\theta, \lambda)$ by working the gradient vector of $f(x, y, z) = x^2 + y^2 - z^2$ at $\mathbf{r}(\theta, \lambda)$.

(iv) What is the tangent plane to the hyperboloid at $\mathbf{r}(\theta, \lambda)$?

3. (i) What is the shortest distance between $\left(0, 1/\sqrt{2}, 1/\sqrt{2}\right)$ and $\left(1/2, 0, \sqrt{3}/2\right)$ as measured on the unit sphere $x^2 + y^2 + z^2 = 1$?

(ii) Let $\gamma(t) = (\cos\theta(t), \sin\theta(t), z(t))$, where $a \leqslant t \leqslant b$, be a parametrized curve in the cylinder $x^2 + y^2 = 1$. Show that the curve γ has arc length

$$\int_a^b \sqrt{\dot\theta^2 + \dot z^2}\,\mathrm{d}t.$$

Deduce that the map $(\theta, z) \to (\cos\theta, \sin\theta, z)$ from $\mathbb{R}^2$ to the cylinder is an isometry (when distances are measured on the cylinder).

Find the shortest distance from $(1, 0, 0)$ to $(0, 1, 1)$ when measured on the cylinder.

4. Let S denote the unit sphere $x^2 + y^2 + z^2 = 1$. Every point $P = (r\cos\alpha, r\sin\alpha)$ in $\mathbb{R}^2$ can be identified with a point Q on S by drawing a line from $(r\cos\alpha, r\sin\alpha, 0)$ to the sphere's north pole $N = (0, 0, 1)$; this line intersects the sphere at two points Q and N. We define a map f from $\mathbb{R}^2$ to the sphere S by setting $f(P) = Q$.

(i) Show that

$$Q = \left(\frac{2r\cos\alpha}{1 + r^2}, \frac{2r\sin\alpha}{1 + r^2}, \frac{r^2 - 1}{1 + r^2}\right).$$

(ii) What are the spherical polar co-ordinates θ, ϕ of the point Q?

5. (i) Let $\mathbf{r}(u, v)$ be a parametrized surface with unit normal $\mathbf{n}(u, v)$. By differentiating $\mathbf{n} \cdot \mathbf{n} = 1$, show that $\mathbf{n}_u$ and $\mathbf{n}_v$ are tangent vectors to the surface. Deduce that

$$\mathbf{n}_u \wedge \mathbf{n}_v = K(u, v)\,\mathbf{r}_u \wedge \mathbf{r}_v$$

for some scalar function $K(u, v)$.

(ii) When $\mathbf{r}(u, v)$ is the sphere of radius R centred at $\mathbf{0}$, so that $\mathbf{r} = R\mathbf{n}$, what is $K(u, v)$?

(iii) Let $0 < a < b$. Consider the following parametrization of a torus

$$\mathbf{r}(\theta, \phi) = ((b + a\cos\theta)\cos\phi, (b + a\cos\theta)\sin\phi, a\sin\theta).$$

Determine $\mathbf{n}(\theta, \phi) = \mathbf{r}_\theta \wedge \mathbf{r}_\phi / \left|\mathbf{r}_\theta \wedge \mathbf{r}_\phi\right|$ and hence find $K(\theta, \phi)$. Where is $K(\theta, \phi)$ positive, where negative?

6. (*Optional*) Suppose that two planar co-ordinate systems (x, y) and (X, Y) are related by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$

where $ad \neq bc$ and let $f(x, y) = F(X, Y)$. Show that Laplace's equation is invariant – that is

$$F_{XX} + F_{YY} = 0 \qquad \Longleftrightarrow \qquad f_{xx} + f_{yy} = 0$$

– if and only if the above 2×2 matrix can be written as λA where$\lambda > 0$ and A is an orthogonal matrix.