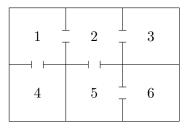
Problem Sheet 5

Please send any comments or corrections to martin@stats.ox.ac.uk.

- 1. A bug jumps around the vertices of a triangle. At every jump, it moves from its current position to either of the other two vertices with probability 1/2 each (independently of how it arrived at its current position). The bug starts at vertex 1. Let p_n be the probabilility that it is at vertex 1 after n jumps. Find the value of p_n for each n. (*Hint: find an appropriate first-order linear recurrence relation.*) What happens to p_n as $n \to \infty$?
- 2. The diagram below shows the floor plan of a house with six rooms: in room 1 is a mouse which will change rooms every minute, first moving at t = 1 and choosing a door to an adjoining room at random. In room 6 is a sleeping but hungry cat which will instantly wake if the mouse should enter. How long on average can we expect the mouse to survive?



3. (Gambler's ruin, symmetric case.) A gambler starts a game with a bankroll of $\pounds n$ where $n \in \{1, 2, \ldots, M-1\}$. At each step of the game, he wins $\pounds 1$ with probability 1/2 and loses $\pounds 1$ with probability 1/2, independently for different steps. The game ends when the gambler's bankroll reaches $\pounds 0$ or $\pounds M$.

In lectures we saw that the probability the gambler finishes with $\pounds M$ is n/M.

- (a) What is the expected amount of money that the gambler has at the end of the game?
- (b) Suppose we know that the gambler ends the game with $\pounds M$. What is the conditional probability that he won $\pounds 1$ on the first step?
- (c) Let e_n be the expected length of the game. Find e_n for each n. For which n is e_n largest?
- 4. (a) Suppose that X has a geometric distribution with parameter p. Show that the probability generating function of X is

$$G_X(s) = \frac{ps}{1 - (1 - p)s}, \quad \text{for } |s| < \frac{1}{1 - p}.$$

- (b) Use this to calculate the mean and variance of X.
- 5. (a) A fair coin is tossed n times. Let r_n be the probability that the sequence of tosses never has a head followed by a head. Show that

$$r_n = \frac{1}{2}r_{n-1} + \frac{1}{4}r_{n-2}, \quad n \ge 2.$$

Find r_n using the conditions $r_0 = r_1 = 1$. Check that the value you get for r_2 is correct.

- (b) Let X be the number of coin tosses needed until you first get two heads in a row. (Note that $X \ge 2$.) Find the probability mass function of X.
- (c) Find the probability generating function of X. Use this to calculate the mean of X. (You may wish to check that your answer agrees with what you got for Q6 on Problem Sheet 3!)
- (d) Let Y be the number of coin tosses needed until you first see a tail followed by a head. On any two particular coin tosses, the probability of seeing the pattern TH is 1/4, the same as the probability of seeing the pattern HH. So $\mathbb{P}(Y = 2) = \mathbb{P}(X = 2) = 1/4$. Find $\mathbb{P}(Y > n)$ for $n \ge 1$ and compare it to $\mathbb{P}(X > n)$. Is your answer surprising?
- 6. (Optional. If you liked the coupon collector problem on sheet 3, you may enjoy this question too!)

Consider a symmetric random walk on a cycle with N sites, labelled 0, 1, 2, ..., N - 1. A particle starts at site 0, and at each step it jumps from its current site *i* to one of its two neighbours $i + 1 \mod N$ and $i - 1 \mod N$ with equal probability (independently of how it arrived at its current position).

- (a) Find the expected number of steps until every site has been visited. [Hint: just after a new site has been visited, what does the set of visited sites look like? The value $e_1 = M - 1$ from question 3(c) may be useful!]
- (b) For each k = 1, ..., N-1, what is the probability that k is the last site to be visited? [*Hint: before visiting site k, the walk must visit either site* k-1 *or site* k+1*. What needs to happen from that point onwards?*]