

## Problem Sheet 8

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1. Continuous random variables  $X$  and  $Y$  have joint probability density function

(a)  $f_{X,Y}(x,y) = C_1(x^2 + \frac{1}{3}xy)$ ,  $x \in (0,1)$ ,  $y \in (0,2)$ .

(b)  $f_{X,Y}(x,y) = C_2e^{-x-y}$ ,  $0 < x < y < \infty$ .

Find the values of the constants  $C_1$  and  $C_2$ . For each of the joint densities above:

- are  $X$  and  $Y$  independent?
- find the marginal probability density functions of  $X$  and of  $Y$ .
- find  $\mathbb{P}(X \leq 1/2, Y \leq 1)$ .

In case (b), if the region had been  $0 < x, y < \infty$ , how would this affect your answer to the question about independence?

2. In the game of Oxémon Ko, you wander the streets of an old university town in search of a set of  $n$  different small furry creatures.

Let  $T_i$  be the time (in hours) at which you first see a creature of type  $i$ , for  $1 \leq i \leq n$ . Suppose that  $(T_i, 1 \leq i \leq n)$  are independent, and that  $T_i$  has exponential distribution with parameter  $\lambda_i$ .

- (a) Let  $X = \min\{T_1, T_2, \dots, T_n\}$  be the time at which you see your first creature. Show that  $X$  has an exponential distribution and give its parameter. [*Hint: consider  $\mathbb{P}(X > t)$  and use independence.*]
- (b) What is the expected number of types of creature that you have not met by time 1?
- (c) Let  $M = \max\{T_1, T_2, \dots, T_n\}$  be the time until you have met all  $n$  different types of creature. Suppose now they are all equally common, with  $\lambda_i = 1$  for all  $i$ . Find the median of the distribution of  $M$ . (As well as giving an exact expression, try to describe how quickly it grows as  $n$  becomes large.) [*Here you may wish to consider instead  $\mathbb{P}(M \leq t)$ . You may find useful an estimate like  $\alpha^{1/n} - 1 = e^{\frac{1}{n} \log \alpha} - 1 \approx \frac{1}{n} \log \alpha$  for large  $n$ .]*]

3. Let  $U$  and  $V$  be independent random variables, both uniformly distributed on  $[0, 1]$ . Find the probability that the quadratic equation  $x^2 + 2Ux + V = 0$  has two real solutions.
4. A fair die is thrown  $n$  times. Using Chebyshev's inequality, show that with probability at least  $31/36$ , the number of sixes obtained is between  $n/6 - \sqrt{n}$  and  $n/6 + \sqrt{n}$ .
5. Suppose that you take a random sample of size  $n$  from a distribution with mean  $\mu$  and variance  $\sigma^2$ . Using Chebyshev's inequality, determine how large  $n$  needs to be to ensure that the difference between the sample mean and  $\mu$  is less than two standard deviations with probability exceeding 0.99.

6. A fair coin is tossed  $n + 1$  times. For  $1 \leq i \leq n$ , let  $A_i$  be 1 if the  $i$ th and  $(i + 1)$ st outcomes are both heads, and 0 otherwise.
- (a) Find the mean and the variance of  $A_i$ .
  - (b) Find the covariance of  $A_i$  and  $A_j$  for  $i \neq j$ . (Consider the cases  $|i - j| = 1$  and  $|i - j| > 1$ .)
  - (c) Define  $M = A_1 + \dots + A_n$ , the number of occurrences of the motif HH in the sequence. Find the mean and variance of  $M$ . [Recall the formula for the variance of a sum of random variables, in terms of their variances and pairwise covariances.]
  - (d) Use a similar method to find the mean and variance of the number of occurrences of the motif TH in the sequence.
7. Let  $a, b, p \in (0, 1)$ . What is the distribution of the sum of  $n$  independent Bernoulli random variables with parameter  $p$ ? By considering this sum and applying the weak law of large numbers, identify the limit

$$\lim_{n \rightarrow \infty} \sum_{\substack{r \in \mathbb{N}: \\ an < r < bn}} \binom{n}{r} p^r (1-p)^{n-r}$$

in the cases (i)  $p < a$ ; (ii)  $a < p < b$ ; (iii)  $b < p$ .