Bayes' theorem:

$$\boxed{\mathbb{P}(B|A) = \frac{\mathbb{P}(A|B)\mathbb{P}(B)}{\mathbb{P}(A|B)\mathbb{P}(B) + \mathbb{P}(A|B^c)\mathbb{P}(B^c)}}$$

Example: medical test for a disease.

$$B = \{\text{patient has disease}\}$$
$$A = \{\text{patient tests positive}\}.$$

Suppose $\mathbb{P}(A|B) = 1$, and $\mathbb{P}(A|B^c) = 0.01$ (false positive rate).

What is $\mathbb{P}(B|A)$? (i.e. the conditional probability of the disease being present, given a positive test)

To answer, we need to know $\mathbb{P}(B)$.

We will see that $\mathbb{P}(B^c|A)$ can be large, even when $\mathbb{P}(A|B^c)$ is small.

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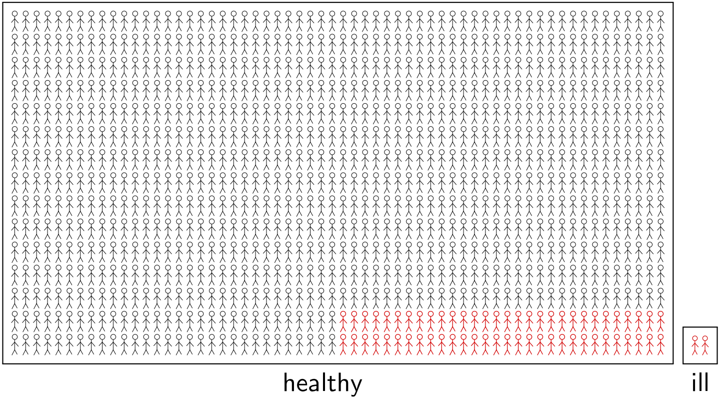

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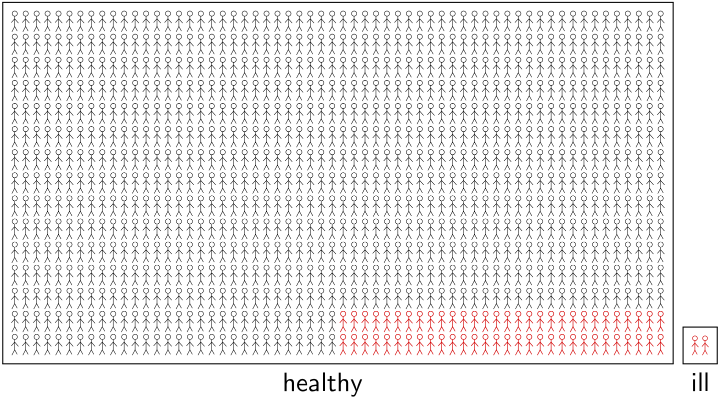

Even if $\mathbb{P}(A|B^c)$ is small, $\mathbb{P}(B^c|A)$ can be large (if $\mathbb{P}(B)$ is tiny).
(See example sheet 2)

Prosecutor's fallacy

- A defendant is on trial. A DNA test shows a match between the defendant and material left at the scene of the crime.
- Prosecutor: "the chance of a false positive match is 1 in a million".
- The jury may understand: "the chance that the defendant is innocent is 1 in a million".

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- But we could interpret another way. In a population of 65 million people, there are about 65 people who would give a DNA match. The defendant is one of these 65 people.
- The DNA test may be useful *additional* evidence if other relevant information already links the defendant to the crime. But if the match was found e.g. by searching through a pre-existing DNA database, the match may be highly unreliable.

Simpson's paradox

Comparison of surgical procedures for kidney stones (Charig et al, 1986).

	Number	Success rate
Treatment A (open surgery)	350	(273/350 =) 0.78
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One can divide the patients into two groups, according to size of stones:

	Type I (stone $<$ 2cm)		Type II (stone $>$ 2cm)	
	Number	Success rate	Number	Success rate
Treatment A	87	(81/87 =) 0.93	263	(192/263 =) 0.73
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Simpson's paradox: we can have

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and yet $$\mathbb{P}(E|F) < \mathbb{P}(E|F^c).$$

Simpsons paradox

Each day for a week, Marge makes Bart and Lisa sit down and answer questions from a maths quiz book before breakfast.

Every day, Lisa gets a higher proportion of the questions she answers right than Bart does out of the questions he answers.

But overall for the week, Bart has a higher proportion of correct answers than Lisa.

How come?!