Prelims Introductory Calculus MT 2019: Sheet 1

1. Evaluate the following integrals:

$$\text{(a)} \int_0^1 \frac{1}{e^x+1}\,\mathrm{d}x, \qquad \text{(b)} \int_0^2 x^3 e^{-x}\mathrm{d}x, \qquad \text{(c)} \int (2x^3 - x)\tan^{-1} x \,\mathrm{d}x.$$

2. Let

$$I_n = \int_0^\infty x^n e^{-x^2}\,\mathrm{d}x.$$

Show that

$$I_n = \frac{n-1}{2} I_{n-2}, \qquad n \geq 2.$$

Find I_5 and, given that $I_0 = \sqrt{\pi}/2$, find I_6.

3. Solve the initial-value problems

(a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy^2 + x}{x^2 y + y}, \quad y(\sqrt{2}) = 1,$$

(b)

$$\sin x \sin y \frac{\mathrm{d}y}{\mathrm{d}x} = \cos x \cos y, \quad y\left(\frac{\pi}{2}\right) = \frac{\pi}{4}.$$

4. Solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x+y+2}, \qquad y(0) = 1,$$

using the substitutions

$$X = x + a \quad \text{and} \quad Y = y + b,$$

for some suitable constants a and b which you should find.

5. Find the solution of the initial value problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{1}{1+x^2}, \qquad y(0) = y'(0) = 0,$$

and verify that $y(1) = \frac{\pi}{4} - \frac{1}{2}\ln 2$.

6. By making a substitution $z = \mathrm{d}y/\mathrm{d}x$, or otherwise, solve the initial value problem

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = (1+3x^2)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \qquad y(1) = 0, \;\; y'(1) = -\frac{1}{2}.$$