Prelims Introductory Calculus MT 2019: Sheet 3

1. Let
$$f(x, y) = \exp\left(\frac{y}{x}\right),$$
where $x \neq 0$. Find all the first order and second order partial derivatives of f.

2. The polar co-ordinates r and θ are defined for $x > 0$, $y \in \mathbb{R}$ by
$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$
   (a) Sketch the curves $r =$ const. and $\theta =$ const. Using your sketches, and without calculating any partial derivatives, determine the points at which $\frac{\partial r}{\partial y}$ is positive. At what points is $\frac{\partial y}{\partial r}$ positive?
   (b) Find partial derivatives
$$\frac{\partial r}{\partial x}, \frac{\partial \theta}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial y}, \frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}.$$
   Verify that $\frac{\partial r}{\partial y}\frac{\partial y}{\partial r} < 1$ at all points.

3. Let $F(x, t) = f(x - ct) + g(x + ct)$, where f and g are differentiable functions of one variable, and c is a constant. Show that
$$c^2 \frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial t^2}.$$

4. Let $F(x, y) = f(y \ln x)$ where f is a twice differentiable function of one variable and $x > 0$. Show that
$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = xy\frac{\partial^2 F}{\partial x \partial y} - x^2 \ln x \frac{\partial^2 F}{\partial x^2}.$$

5. (a) Suppose that F is a differentiable function of x and y, and that y is a function of x. Show that
$$\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}.$$
   If $y(x)$ is defined implicitly by the equation $F(x, y) = 0$, deduce that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial y},$$
   provided that $\dfrac{\partial F}{\partial y} \neq 0$.
   (b) Let $F(x, y) = f(x^2 + g(x + 2y))$, where f and g are differentiable functions of one variable. Given that the equation $F(x, y) = 0$ defines a function $y(x)$, show that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2x + g'(w)}{2g'(w)} \quad \text{and} \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{(g'(w))^2 + 2x^2 g''(w)}{(g'(w))^3}$$
   where $w = x + 2y$. State any restrictions that are needed on f and g.

6. The variables u and v are given in terms of x and y by
$$u = x^2 - y^2 \quad \text{and} \quad v = 2xy.$$
   Let $g(u, v) = f(x, y)$ be differentiable functions of two variables.
   (a) Use the chain rule to show that
$$\frac{\partial^2 f}{\partial x^2} = 2\frac{\partial g}{\partial u} + 4x^2\frac{\partial^2 g}{\partial u^2} + 8xy\frac{\partial^2 g}{\partial u \partial v} + 4y^2\frac{\partial^2 g}{\partial v^2},$$
   and find a similar expression for $\partial^2 f/\partial y^2$.
   (b) Hence express $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2$ in terms of the partial derivatives of g, and deduce that if $f(x, y) = x + y$ then
$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = 0.$$