Prelims Introductory Calculus MT 2019: Sheet 5

1. Calculate the Jacobians $\dfrac{\partial(u,v)}{\partial(x,y)}$ and $\dfrac{\partial(x,y)}{\partial(u,v)}$, and verify that $\dfrac{\partial(u,v)}{\partial(x,y)}\dfrac{\partial(x,y)}{\partial(u,v)} = 1$, in each of the following cases:

   (i) $u = x + y, \quad v = \dfrac{y}{x}$; (ii) $u = \dfrac{x^2}{y}, \quad v = \dfrac{y^2}{x}$.

2. The variables u and v are given by

$$u = x^2 - xy, \qquad v = y^2 + xy$$

   for all real x and y. By finding an appropriate Jacobian matrix, calculate the partial derivatives x_u, x_v, y_u and y_v in terms of x and y only. State the values of x and y for which your results are valid.

3. Recall the definition of parabolic coordinates:

$$x = \frac{1}{2}(u^2 - v^2), \qquad y = uv.$$

   Show that Laplace's equation in Cartesian coordinates, that is

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0,$$

   transforms into the same equation in parabolic coordinates.

4. In the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} - 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = 0,$$

   make the change of variables $s = y + 2x, \ t = y + 3x$ and show that the PDE becomes

$$\frac{\partial^2 z}{\partial s \partial t} = 0.$$

   Hence solve the original PDE.

5. Show that if $x = r\cos\theta, \ y = r\sin\theta$, the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{become} \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

   (These equations, named the Cauchy-Riemann equations, are fundamental in complex analysis. Notice that the functions u and v as above automatically satisfy Laplace's equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$.)