Prelims Introductory Calculus MT 2019: Sheet 7

1. Find the gradient ∇f when

   (a) $f(x,y) = e^{x^2-y^2}\sin 2xy$; (b) $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$.

2. For each of the following f, sketch some contours $f =$ constant and indicate ∇f by arrows at some points on these curves. Use one set of axes per f.

   (a) $f(x,y) = xy$; (b) $f(x,y) = \dfrac{1}{4}x^2 + y^2$; (c) $f(x,y) = \dfrac{x}{y}$.

3. Show that there is no function $f(x,y,z)$ such that $\nabla f(x,y,z) = (y,z,x)$.

4. Let $f(x,y) = x^2y^3$. What is ∇f? Determine

   $$\lim_{t\to 0} \frac{f(\mathbf{a}+t\mathbf{v}) - f(\mathbf{a})}{t},$$

   where $\mathbf{a} = (1,1)$ and $\mathbf{v} = (a,b)$ is a unit vector such that $a^2+b^2 = 1$. What is the maximum value of the limit that can be obtained by varying the vector $\mathbf{v}$?

5. A bug in the xy-plane finds itself in a toxic environment. The level of toxicity is given by the function $f(x,y) = 2x^2y - 3x^3$. The bug is at the point $(1,2)$. In what direction away from $(1,2)$ should it initially move in order to lower its exposure to the toxin as rapidly as possible?

6. Find a unit vector perpendicular to the surface $x^2y + y^2z + z^2x + 1 = 0$ at the point $(1,2,-1)$. Write down equations of the tangent plane and the normal line at this point.

7. Let f and g be sufficiently differentiable functions of x, y, z.

   (a) Prove that

      i. $\nabla(fg) = f\nabla g + g\nabla f$;

      ii. $\nabla(f^n) = nf^{n-1}\nabla f$;

      iii. $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$;

   (b) The Laplacian operator ∇^2 is defined by

   $$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

   Show that

   $$\nabla^2(fg) = f\nabla^2 g + 2(\nabla f)\cdot(\nabla g) + g\nabla^2 f.$$

8. Use the Taylor series for a function of two variables to expand the following functions about $x = 1, y = 1$, to the order shown.

   (a) $f(x,y) = x^3 + y^2 + 3x^2y$, to all orders;

   (b) $f(x,y) = x^2 + y + \cos(2\pi xy)$, to second order.