Prelims Introductory Calculus MT 2019: Sheet 8

1. Find and classify the critical points of

   (a)
   $$f(x,y) = 6x^2 - 2x^3 + 3y^2 + 6xy,$$

   (b)
   $$f(x,y) = xy(x^2 + y^2 - 1),$$

   (c)
   $$f(x,y) = \sin^2 x + \sin^2 y - \cos^2 x \cos^2 y.$$

2. Let
   $$f(x,y) = x^3 + x^2 + 2\alpha xy + y^2 + 2\alpha x + 2y,$$
   where α is a positive constant. Find and classify the critical points of $f(x,y)$
   (a) when $\alpha > 1$, (b) when $0 < \alpha < 1$.

3. Find the volume of the largest cuboid (i.e. box) with edges parallel to the axes, inscribed in the ellipsoid
   $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

4. Heron's formula for the area A of a triangle with sides of length $a, b, c > 0$ is
   $$A = \sqrt{s(s-a)(s-b)(s-c)}, \quad \text{where } s = \frac{a+b+c}{2}.$$
   Use the method of Lagrange multipliers to answer the following:
   (a) Show that for a given fixed perimeter $P = 2s$ of the triangle, the area is maximised when the triangle is equilateral. What is the maximum area?
   (b) Find, in terms of the fixed perimeter $P = 2s$, the maximum area of a right-angled triangle with perimeter P.

5. Use the method of Lagrange multipliers to find the shortest distance from the origin to the line of intersection of the planes $2x + y - z = 1$ and $x - y + z = 2$.
   Optional (for those taking Geometry): Repeat the calculation using geometric methods.

6. Let n be a positive integer and let $a > 0$, $b > 0, c > 0$ and $r > 0$ be real constants.
   Show that the stationary value of the function
   $$f(x,y,z) = \left(\frac{a}{x}\right)^n + \left(\frac{b}{y}\right)^n + \left(\frac{c}{z}\right)^n, \quad x > 0,\ y > 0, z > 0,$$
   subject to the constraint $x^2 + y^2 + z^2 = r^2$ is given by
   $$x = \frac{ra^{n/(n+2)}}{A}, \quad y = \frac{rb^{n/(n+2)}}{A}, \quad z = \frac{rc^{n/(n+2)}}{A},$$
   where $A = \left(a^{2n/(n+2)} + b^{2n/(n+2)} + c^{2n/(n+2)}\right)^{1/2}$.