ANALYSIS I: Problem sheet 3

Sequences: Convergence, Infinite Limits, Complex Sequences

[*Where needed you may assume properties of familiar functions, even though they have not been rigorously defined yet in the course.*]

1. For each of the following choices of a_n, and for arbitrary $\varepsilon > 0$, find N such that $|a_n| < \varepsilon$ whenever $n \geqslant N$. [*You only need to find a value of N that works and not necessarily the smallest such N.*]

$$\text{(i)}\quad \frac{1}{n^2+3}, \qquad \text{(ii)}\quad \frac{1}{n(n-\pi)}, \qquad \text{(iii)}\quad \frac{1}{\sqrt{5n-1}}.$$

2. Use sandwiching arguments to prove that, for each of the following choices of a_n, the sequence (a_n) converges to 0:

$$\text{(i)}\quad \frac{n+1}{n^2+n+1}, \qquad \text{(ii)}\quad 2^{-n}\cos(n^2), \qquad \text{(iii)}\quad \sin\frac{1}{n}, \qquad \text{(iv)}\quad \begin{cases} 1/2^n & \text{if } n \text{ is prime,} \\ -1/3^n & \text{otherwise.} \end{cases}$$

3. (a) Prove that $\sqrt{n+1} - \sqrt{n} \to 0$ by using the identity $a - b = (\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$ (for $a, b \in \mathbb{R}^{\geqslant 0}$).

   (b) Prove that $n^{1/n} \geqslant 1$ for $n = 1, 2, \ldots$. Let $a_n = n^{1/n} - 1$. Prove, by applying the binomial theorem to $(1 + a_n)^n$, that

$$a_n \leqslant \sqrt{\frac{2}{n-1}} \quad \text{for } n > 1.$$

   Deduce that $n^{1/n} \to 1$.

4. (a) Write down, carefully quantified, what it means for a real sequence (a_n) (i) to be convergent; (ii) to be bounded; (iii) to tend to infinity. Write down, carefully quantified, the negations of (i), (ii), (iii).

   (b) Formulate and prove an analogue of the Sandwiching Lemma applicable to real sequences which tend to infinity.

   (c) For each of the following choices of a_n decide whether or not the sequence tends to infinity:

$$\text{(i)}\quad \frac{n^2+n+1}{n+1}, \qquad \text{(ii)}\quad n^2 \sin n, \qquad \text{(iii)}\quad \frac{n^{3/4}}{\sqrt{5n-1}}, \qquad \text{(iv)}\quad \left(1+\frac{1}{n}\right)^n.$$

   Justify your answers briefly.

turn over/ ...

5. Assume that (a_n) is a sequence such that $a_n \to 0$. Let (b_n) be a bounded sequence. Prove that $a_n b_n \to 0$.

   Give an example of a single sequence (a_n) such that $a_n \to 0$ and of appropriate sequences (c_n) to demonstrate that each of the following possibilities can occur:

   (i) $a_n c_n \to 0$ and (c_n) is unbounded;

   (ii) $a_n c_n \to \infty$;

   (iii) $(a_n c_n)$ converges to a non-zero limit;

   (iv) $(a_n c_n)$ is bounded and divergent;

   (v) $a_n c_n \to -\infty$.

6. For each of the following choices of z_n decide whether or not (z_n) converges:

$$\text{(i)}\quad \left(\frac{1}{1+\mathrm{i}}\right)^n, \qquad \text{(ii)}\quad \frac{(1-\mathrm{i})n}{n+\mathrm{i}}, \qquad \text{(iii)}\quad (-1)^n \frac{n+\mathrm{i}}{n}.$$

   Give brief justifications for your answers.

7. [Later parts may be treated as optional] Let c be a complex number. The complex numbers $z_n(c)$ are defined recursively by

$$z_1(c) = c, \qquad z_{n+1}(c) = (z_n(c))^2 + c \quad \text{for} \quad n \geqslant 1.$$

   The *Mandelbrot set* is defined by

$$M = \{\, c \in \mathbb{C} \mid \text{ the sequence } (z_n(c)) \text{ is bounded} \,\}.$$

   (a) Show that each of $-2, -1, 0, \mathrm{i}$ lies in M but that $1 \notin M$.

   (b) Show that if $c \in M$ then $\overline{c} \in M$, where $\overline{c}$ denotes the conjugate of c.

   (c) Show that if $|c| \leqslant 1/4$ then $|z_n(c)| < 1/2$ for all n. (So if $|c| \leqslant 1/4$ then $c \in M$.)

   (d) Show that if $|c| = 2 + \varepsilon$ where $\varepsilon > 0$, then $|z_n(c)| \geqslant 2 + a_n\varepsilon$ for $n \geqslant 1$ where $a_n = (4^n + 2)/6$. Deduce that the Mandelbrot set lies entirely within the disc $|z| \leqslant 2$.

Points to ponder

A. Let

$$a_n = 1 + \frac{2^{n/1000} - 1}{n^2} \quad \text{and} \quad b_n = \frac{n^2 \cos n}{2^{n/1000} - 1}.$$

   Use a calculator to estimate a_n and b_n for $n = 1, 2, 5, 10, 100, 1000$. What do you think is the behaviour of a_n and b_n as n tends to infinity? What confidence do you have in your answers?

B. Why can you not analyse the limiting behaviour of $(n^{1/n})$ by using the first principles argument applicable to (n^α), where α is a constant?