ANALYSIS I: Problem sheet 4

Subsequences, Algebra of Limits, Monotonic Sequences

1. (a) Let the sequence (a_n) be defined by

$$a_n = \left(\frac{n^2-1}{n^2+1}\right)\cos(2\pi n/3).$$

By considering suitable subsequences prove that (a_n) diverges.

(b) Consider the sequence $(\cos n)$. Show that, for a suitable positive constant K, there exist subsequences (b_r) and (c_s) of $(\cos n)$ with $b_r > K$ for all r and $c_s < -K$ for all s. Deduce that $(\cos n)$ diverges.

2. (a) Let (a_n) be a sequence such that the subsequences (a_{2n}) and (a_{2n+1}) both converge to a real number L. Show that (a_n) also converges to L.

(b) Let (b_n) be a sequence such that each of the subsequences (b_{2n}), (b_{2n+1}), (b_{3n}) converges. Need (b_n) converge? Either provide a proof or a counterexample.

(c) Let (c_n) be a sequence such that the subsequence (c_{kn}) converges for each $k = 2, 3, 4, \ldots$. Need (c_n) converge? Provide either a proof or a counterexample.

3. For which of the following choices of a_n does the sequence (a_n) converge? Justify your answers, and find the value of the limit when it exists.

$$\text{(i)}\ \frac{n^2}{n!};\quad \text{(ii)}\ \frac{2^n n^2 + 3^n}{3^n(n+1)+n^7};\quad \text{(iii)}\ \frac{(n!)^2}{(2n)!};\quad \text{(iv)}\ \frac{n^4+n^3\sin n+1}{5n^4 - n\log n}.$$

[*You may freely make use of standard limits and inequalities, sandwiching and AOL methodology, as appropriate.*]

4. (a) Let (a_n) be a real sequence. Prove from the limit definition that $a_n \geqslant 0$ and $a_n \to L$ implies $L \geqslant 0$ and prove further that $\sqrt{a_n} \to \sqrt{L}$.

(b) Let (a_n), (b_n) and (c_n) be sequences of real numbers converging to L_1, L_2, L_3, respectively. Let $d_n = \max\{a_n, b_n, c_n\}$. Assuming any standard AOL results that you require, prove that $d_n \to \max\{L_1, L_2, L_3\}$.

5. Let $r > 0$. Let $a_n = r^n/n!$.

(a) By considering a_{n+1}/a_n show that the tail $(a_n)_{n\geqslant N}$ is monotonic decreasing if N is sufficiently large. [*You should specify a suitable value of* N.]

(b) Show that (a_n) converges to a limit L and find the value of L.

6. The real sequence (a_n) is defined by

$$a_1 = c, \qquad (\alpha+\beta)a_{n+1} = a_n^2 + \alpha\beta,$$

where $0 < \alpha < \beta$ and $c > \alpha$.

(a) Prove that if (a_n) converges to a limit L then necessarily $L = \alpha$ or $L = \beta$.

(b) Prove that $a_{n+1} - \gamma$ and $a_n - \gamma$ have the same sign, where γ denotes either α or β.

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(c) Prove that, if $c < \beta$ then (a_n) converges monotonically to α. Discuss the limiting behaviour of (a_n) when $c \geqslant \beta$.

(d) Prove that, if $\alpha < c < \beta$,

$$|a_n - \alpha| \leqslant \left(\frac{\alpha + c}{\alpha + \beta}\right)^{(n-1)} (c - \alpha).$$

7. [Optional, to provide additional practice with sequences defined by recurrence relations] Let (a_n) be the sequence of real numbers given by

$$a_1 = a, \quad a_{n+1} = \frac{2}{a_n + 1} \quad (n \geqslant 1).$$

(a) Assume $0 < a < 1$. Prove that the subsequences (a_{2n}) and (a_{2n+1}) are monotonic, one increasing and the other decreasing. Prove that each of these subsequences converges, and find their limits. Deduce that (a_n) converges.

(b) What happens if $a > 1$?

Points to ponder

A. [Infinite limits and results of AOL type] Let (a_n) and (b_n) be real sequences and let $c_n = a_n + b_n$ and $d_n = a_n - b_n$. Consider the scenarios

(a) $a_n \to L \in \mathbb{R}$ and $b_n \to \infty$;

(b) $a_n \to \infty$ and $b_n \to M \in \mathbb{R}$;

(c) $a_n \to \infty$ and $b_n \to \infty$;

What are the possible limiting behaviours of (c_n) and (d_n) in each case? Find examples to illustrate the possibilities and proofs of any assertions which hold in general.

B. Let $x \in \mathbb{R}$. Let

$$a_{m,n} = \cos\left(\frac{m}{n}\pi x\right).$$

What can you say about the iterated limit

$$\lim_{m\to\infty}\left(\lim_{n\to\infty} a_{m,n}\right)?$$

Now consider

$$\lim_{n\to\infty}\left(\lim_{m\to\infty} a_{m,n}\right).$$

Does this iterated limit always exist? Exist for some values of x but not others? Never exist?

What conclusions do you draw?