## ANALYSIS I: Problem sheet 5

Cauchy Sequences, Series (convergence and absolute convergence),
Properties of e, Alternating Series Test.

1. For  $n \in \mathbb{N}$  let

$$a_n = \int_1^n \frac{\cos x}{x^2} \, \mathrm{d}x.$$

Prove, for  $m \ge n \ge 1$ , that  $|a_m - a_n| \le n^{-1}$  and deduce that  $(a_n)$  converges. By integration by parts, or otherwise, demonstrate the existence of

$$\lim_{n \to \infty} \int_{1}^{n} \frac{\sin x}{x} \, \mathrm{d}x.$$

2. (a) Let p be a non-zero natural number. Prove by considering the partial sums that

$$\sum \frac{1}{k(k+p)}$$

converges. What is  $\sum_{k=1}^{\infty} 1/(k(k+p))$ ?

(b) Evaluate the sum

$$\sum_{k=1}^{\infty} \frac{\cos k}{2^k}.$$

(c) Use the Comparison Test to prove that

$$\sum \frac{2k+1}{(k+1)(k+2)^2}$$

converges.

3. Let  $(a_n)$  be a sequence of real or complex numbers and assume that  $\sum_{k=1}^{\infty} |a_k|$  converges. Prove, by considering the partial sums

$$s_n = a_1 + \dots + a_n$$
 and  $S_n = |a_1| + \dots + |a_n|$ ,

that

$$\left| \sum_{k=1}^{\infty} a_k \right| \leqslant \sum_{k=1}^{\infty} |a_k|.$$

[You may assume the Triangle Law for real or complex numbers and the theorem that  $\sum |a_k|$  converges implies that  $\sum a_k$  converges.]

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4. The number known as e is defined by

$$e = \lim_{n \to \infty} s_n$$
, where  $s_n = \sum_{k=0}^n \frac{1}{k!}$ .

- (a) Prove that  $(s_n)$  is monotonic increasing and bounded above and deduce that the limit defining e exists. [Hint for getting an upper bound: compare  $s_n$  with the sum of a geometric progression.]
- (b) Show that, for  $n \ge 1$ ,

$$0 < e - \sum_{k=0}^{n} \frac{1}{k!} < \frac{1}{n! \, n}.$$

Deduce that e is irrational.

5. There exists a real number L such that

$$\left(1 + \frac{1}{n}\right)^n \to L \quad \text{as } n \to \infty$$

(recall Example 9.3(a)). In fact L = e [a proof is given in the supplementary notes on e]. Assuming this fact, show that

$$\left(1 - \frac{1}{n}\right)^n \to \frac{1}{e} \quad \text{as } n \to \infty.$$

- 6. (a) Prove that  $\sum (-1)^{k-1}(\sqrt{k+1}-\sqrt{k})$  converges.
  - (b) Let

$$s_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots + \frac{(-1)^{n+1}}{2n-1}.$$

The Alternating Series Test implies that  $(s_n)$  converges to some limit L. By examining the proof of the AST prove that 2/3 < L < 13/15.

- 7. Let  $\sum a_k$  be a series of real numbers. Which of the following are true and which are false? Provide a proof or counterxample as appropriate.
  - (a)  $k^2 a_k \to 0$  implies  $\sum a_k$  converges.
  - (b) If  $\sum a_k$  converges, then  $\sum (a_k)^2$  converges.
  - (c) If  $\sum a_k$  converges absolutely, then  $\sum (a_k)^2$  converges.
  - (d)  $\sum (a_k)^2$  convergent implies  $\sum (a_k)^3$  convergent.
- 8. [Optional, and quite challenging] For each of the following statements either provide a proof or a counterexample.
  - (a) For a divergent series  $\sum a_k$  of positive terms,  $\sum \frac{a_k}{1+a_k}$  is also divergent.
  - (b) Assume  $a_k > 0$ . Then  $\sum a_k$  and  $\sum a_k/s_k$  either both converge or both diverge, where  $s_k = a_1 + \cdots + a_k$ .

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## Points to ponder

- A. Give an example of a real sequence  $(c_n)$  for which  $|c_{n+1} c_n| \to 0$  as  $n \to \infty$  but for which  $(c_n)$  fails to converge. What does the existence of such an example tell you?
- B. Let  $(a_k)$  be a sequence. Define new sequences  $(b_k)$  and  $(c_k)$  as follows:

$$b_k = (a_{2k-1} + a_{2k}) \quad \text{for } k \geqslant 1;$$
  
$$c_1 = a_1, \quad c_k = (a_{2(k-1)} + a_{2k-1}) \quad \text{for } k \geqslant 2.$$

Now let  $a_k = (-1)^k$  for each  $k \ge 1$ . Calculate

$$s_n := a_1 + a_2 + \dots + a_n;$$
  
 $t_n := b_1 + b_2 + \dots + b_n;$   
 $u_n := c_1 + c_2 + \dots + c_n.$ 

What are the limiting behaviours of  $(s_n)$ ,  $(t_n)$  and  $(u_n)$ ? What happens for other choices of  $(a_k)$ ? What do your answers tell you about infinite sums? Specifically, do such sums always behave in the same way as finite sums do?