Linear Algebra I, Sheet 1, MT2019

Systems of linear equations; matrices and their algebra.

Main course

1. Use any method you can think of to decide which (if any) of the following systems of linear equations with real coefficients have no solutions, which have a unique solution (in which case, what is it?), which have infinitely many solutions.

$$\text{(a)} \left\{ \begin{array}{rcl} 2x+4y-3z &=& 0 \\ x-4y+3z &=& 0 \\ 3x-5y+2z &=& 1 \end{array} \right. ; \quad \text{(b)} \left\{ \begin{array}{rcl} x+2y+3z &=& 0 \\ 2x+3y+4z &=& 1 \\ 3x+4y+5z &=& 2 \end{array} \right. ; \quad \text{(c)} \left\{ \begin{array}{rcl} x+2y+3z &=& 0 \\ 2x+3y+4z &=& 2 \\ 3x+4y+5z &=& 2 \end{array} \right. .$$

2. Let $A := \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B := \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$, $C := \begin{pmatrix} -1 & 5 \\ 4 & 4 \end{pmatrix}$, $D := \begin{pmatrix} 1 & 4 & -3 \\ 2 & 4 & -2 \end{pmatrix}$, $E := \begin{pmatrix} 1 & 2 \end{pmatrix}$ and $F := \begin{pmatrix} -1 & 5 & -6 \\ 3 & 4 & -1 \end{pmatrix}$. For which pairs $X, Y \in \{A, B, C, D, E, F\}$ is $X - 2Y$ defined? And when it is defined, calculate it.

3. Calculate the following matrix products:

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}; \quad \begin{pmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}; \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}^2 .$$

4. Let A be the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(a) Show that A commutes with $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ if and only if A is diagonal (that is, $b = c = 0$).

(b) Which 2×2 matrices A commute with $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$?

(c) Use the results of (a) and (b) to find the matrices A that commute with every 2×2 matrix.

5. Check that you know the definition of the *transpose* of a matrix (it's in the notes). Let A and B be $m \times n$ matrices, and let C be an $n \times p$ matrix.

(a) Show that $(A + B)^T = A^T + B^T$ and that $(\lambda A)^T = \lambda A^T$ for scalars λ.

(b) Show that $(AC)^T = C^T A^T$.

(c) Suppose that $m = n$ and A^{-1} exists. Show that A^T is invertible and that $(A^T)^{-1} = (A^{-1})^T$.

6. Let A and B denote square matrices with real entries. For each of the following assertions, find either a proof or a counterexample.

(a) $A^2 - B^2 = (A - B)(A + B)$.

(b) If $AB = 0$ then $A = 0$ or $B = 0$.

(c) If $AB = 0$ then A and B cannot both be invertible.

(d) If A and B are invertible then $A + B$ is invertible.

(e) If $ABA = 0$ and B is invertible then $A^2 = 0$.

[Hint: where the assertions are false there are usually counterexamples of size 2×2.]

Starter

S1. Let $A := \begin{pmatrix} -1 & 3 \\ 1 & 2 \\ 7 & -2 \end{pmatrix}$, $B := \begin{pmatrix} 4 & -3 & 1 \\ 3 & -2 & 1 \\ 7 & 0 & 1 \end{pmatrix}$, $C := \begin{pmatrix} 2 & -2 & 3 \\ 0 & 0 & 0 \\ 5 & -4 & 3 \end{pmatrix}$. For which pairs X, $Y \in \{A, B, C\}$ is XY defined? When it is defined, calculate it.

S2. Calculate the following matrix products.

$$\begin{pmatrix} x & y & z & w \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}; \quad \begin{pmatrix} x & y & z & w \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

S3. Prove Proposition 5 from the notes:
Let A, B be invertible $n \times n$ matrices. Then AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

Pudding

P1.

(a) Is every diagonal matrix invertible? What can you say about the inverse of an invertible diagonal matrix?

(b) Is every upper triangular matrix invertible? What can you say about the inverse of an invertible upper triangular matrix?

P2.

(a) What happens if we multiply an upper triangular matrix by an upper triangular matrix?

(b) What if we multiply a lower triangular matrix by a lower triangular matrix?

(c) What if we multiply a lower triangular matrix by an upper triangular matrix?

P3.

(a) Let A and B be orthogonal $n \times n$ matrices. Must their product AB also be orthogonal?

(b) Give an example of a 2×2 orthogonal matrix. Give another example. And another. And another. And another. Can you find all possible 2×2 orthogonal matrices?