Linear Algebra I, Sheet 2, MT2019

More about matrices. Elementary row operations; echelon form of a matrix.
Introduction to vector spaces.

Main course

1. Let J_n be the $n \times n$ matrix with all entries equal to 1. Let α, $\beta \in \mathbb{R}$ with $\alpha \neq 0$ and $\alpha + n\beta \neq 0$. Show that the matrix $\alpha I_n + \beta J_n$ is invertible.

[Hint: note that $J_n^2 = nJ_n$; seek an inverse of $\alpha I_n + \beta J_n$ of the form $\lambda I_n + \mu J_n$ where λ, $\mu \in \mathbb{R}$.]

Find the inverse of $\begin{pmatrix} 3 & 2 & 2 & 2 \\ 2 & 3 & 2 & 2 \\ 2 & 2 & 3 & 2 \\ 2 & 2 & 2 & 3 \end{pmatrix}$.

2. Use EROs to reduce each of the following matrices to echelon form:

$$\text{(a)} \begin{pmatrix} 2 & 4 & -3 & 0 \\ 1 & -4 & 3 & 0 \\ 3 & -5 & 2 & 1 \end{pmatrix}; \quad \text{(b)} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 5 & 2 \end{pmatrix}; \quad \text{(c)} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 2 \\ 3 & 4 & 5 & 2 \end{pmatrix}.$$

3. For each $\alpha \in \mathbb{R}$, find an echelon form for the matrix

$$\begin{pmatrix} 1 & 2 & -3 & -2 & 4 & 1 \\ 2 & 5 & -8 & -1 & 6 & 2 \\ 1 & 4 & -7 & 4 & 0 & \alpha \end{pmatrix}.$$

Use your result either to solve the following system of linear equations over $\mathbb{R}$, or to find values of α for which it has no solution:

$$\left\{ \begin{array}{rcrcrcrcrcl} x_1 & + & 2x_2 & - & 3x_3 & - & 2x_4 & + & 4x_5 & = & 1 \\ 2x_1 & + & 5x_2 & - & 8x_3 & - & x_4 & + & 6x_5 & = & 2 \\ x_1 & + & 4x_2 & - & 7x_3 & + & 4x_4 & & & = & \alpha \end{array} \right.$$

4. Use EROs to find the inverses of each of the following matrices

$$\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}; \qquad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}; \qquad \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

5. (a) Show that if the $m \times n$ matrices A, B can be reduced to the same matrix E in echelon form, then there is a sequence of EROs that changes A into B.

(b) Show that an $n \times n$ real matrix may be reduced to RRE form by a sequence of at most n^2 EROs.

6. (a) Prove from the vector space axioms that if V is a vector space, v, $z \in V$ and $v + z = v$, then $z = 0_V$.

(b) Let $V := \mathbb{R} \times \mathbb{Z}$, the set of all pairs (x, k) where x is a real number and k is an integer. Define addition coordinatewise so that $(x, k) + (y, m) = (x + y, k + m)$, and define scalar multiplication by real numbers λ by the rule $\lambda(x, k) = (\lambda x, 0)$. Which of the vector space axioms are satisfied, and which are not?

Starter

S1. Use EROs to determine whether the following matrices are invertible. For each invertible matrix, find the inverse.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}; \qquad \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 2 & 4 & 6 \end{pmatrix}; \qquad \begin{pmatrix} -2 & 0 & 4 & 3 \\ 1 & 7 & 5 & -6 \\ -3 & 7 & 13 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

S2. Show that the inverse of an ERO is an ERO. (Hint: consider each of the three types of ERO separately.)

S3. Prove Lemma 9 from the lecture notes: let V be a vector space over $\mathbb{F}$. Then there is a unique additive identity element 0_V.

Pudding

P1. Use EROs to explore for which real numbers a, b, c, d the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible.

P2. Pick a matrix from question S1. Can you find any relationships between the rows of the matrix? Or between the columns? Does this relate to whether the matrix is invertible? Try this for each of the matrices from S1.

P3. The two-dimensional plane $\mathbb{R}^2$ is a vector space. Which subsets of $\mathbb{R}^2$ themselves have the structure of a vector space (using the same operations of addition and scalar multiplication as in $\mathbb{R}^2$)?