Linear Algebra I, Sheet 3, MT2019

Vector spaces and subspaces. Spanning sets. Linear independence. Bases.

Main course

1. Show that the set of real sequences (u_n) that satisfy the recurrence relation $u_{n+1} = u_n + u_{n-1}$ (for $n \geqslant 1$) is a real vector space (a subspace of the space of all sequences of real numbers).

Find a basis, and write down the dimension of the vector space.

2. For each of the following vector spaces and each of the specified subsets, determine whether or not the subset is a subspace. That is, in each case, either verify the conditions defining a subspace (or use the subspace test), or show by an example that one of the conditions does not hold.

(a) $V = \mathbb{R}^4$: (i) $\{(a, b, c, d) \in V : a + b = c + d\}$;
(ii) $\{(a, b, c, d) \in V : a + b = 1\}$;
(iii) $\{(a, b, c, d) \in V : a^2 = b^2\}$.

(b) $V = \mathcal{M}_{n\times n}(\mathbb{R})$: (i) the set of upper triangular matrices;
(ii) the set of invertible matrices;
(iii) the set of matrices that are not invertible.

3. Let S be a finite spanning set for a vector space V. Let T be a smallest subset of S that spans V. Show that T is linearly independent, hence a basis of V.

4. (a) Which of the following sets of vectors in $\mathbb{R}^3$ are linearly independent?

$$\text{(i) } \{(1, 3, 0), (2, -3, 4), (3, 0, 4)\}, \qquad \text{(ii) } \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}.$$

(b) Let $V := \mathbb{R}^{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{R}\}$. Which of the following sets are linearly independent in V?
(i) $\{f, g, h\}$ where $f(x) = 5x^2 + x + 1$, $g(x) = 2x + 3$ and $h(x) = x^2 - 1$.
(ii) $\{p, q, r\}$ where $p(x) = \cos^2(x)$, $q(x) = \cos(2x)$ and $r(x) = 1$.

5. (a) Let u, v, w be linearly independent vectors in a vector space V.
(i) Show that $u + v$, $u - v$, $u - 2v + w$ are also linearly independent.
(ii) Are $u + v - 3w$, $u + 3v - w$, $v + w$ linearly independent?

(b) Let $\{v_1, v_2, \ldots, v_n\}$ be a linearly independent set of n vectors in a vector space V. Prove that each of the following sets is also linearly independent:
(i) $\{c_1v_1, c_2v_2, \ldots, c_nv_n\}$ where $c_i \neq 0$ for $1 \leqslant i \leqslant n$;
(ii) $\{w_1, w_2, \ldots, w_n\}$ where $w_i = v_i + v_1$ for $1 \leqslant i \leqslant n$.

6. (a) Let $V_1 := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$. Show that V_1 is a subspace of $\mathbb{R}^n$ and find a basis for it.

(b) Let $V_2 := \{(x_{ij}) \in \mathcal{M}_{n\times n}(\mathbb{R}) : x_{ij} = x_{ji}$ for all relevant $(i, j)\}$. Show that V_2 is a subspace of $\mathcal{M}_{n\times n}(\mathbb{R})$—this is the space of real symmetric matrices—and find a basis for it.

(c) Let $V_3 := \{(x_{ij}) \in \mathcal{M}_{n\times n}(\mathbb{R}) : x_{ij} = -x_{ji}$ for all relevant $(i, j)\}$. Show that V_3 is a subspace of $\mathcal{M}_{n\times n}(\mathbb{R})$—this is the space of real *skew-symmetric* $n \times n$ matrices—and find a basis for it.

Starter

S1. Prove Proposition 14 from the lecture notes. That is, let V be a vector space, and take subspaces $U, W \leqslant V$. Then prove that $U + W \leqslant V$ and $U \cap W \leqslant V$.

S2. For each of the following, give an example or prove that no such example exists, first when $V = \mathbb{R}^3$, and second when $V = \mathcal{M}_{2\times 2}(\mathbb{R})$.

(i) A set of 2 linearly independent vectors in V.

(ii) A set of 3 linearly independent vectors in V.

(iii) A set of 4 linearly independent vectors in V.

(iv) A spanning set of 2 vectors in V.

(v) A spanning set of 3 vectors in V.

(vi) A spanning set of 4 vectors in V.

S3. Let V be the set of polynomials of degree at most 2 with real coefficients. That is, $V = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$. Show that this is a vector space (under the usual polynomial addition and scalar multiplication).

Give a basis B_1 for V. Give another basis B_2 that shares exactly one element with B_1. Give a third basis B_3 that shares no elements with B_1 or B_2.

Pudding

P1. Consider V the vector space of all real sequences. For $k \geqslant 1$, let $e^{(k)}$ be the sequence where all terms are 0 except for a 1 in position k. (So $e^{(1)} = (1, 0, 0, \dots)$ and $e^{(2)} = (0, 1, 0, \dots)$ for example.) Let $S = \{e^{(k)} : k \geqslant 1\}$. Is S linearly independent in V? Does S span V?

P2. Let $V = \mathbb{R}^4$. Let $W = \{(x_1, x_2, x_3, x_4) \in V : x_1 + 2x_2 - x_3 = 0\}$. Show that W is a subspace of V.

What is the dimension of W? Find a basis B_W of W.
Consider the standard basis B_V of V. Is there a subset of B_V that is a basis for W?
Can you add one or more vectors to your basis B_W for W to obtain a basis for V?
Can you generalise?

P3. A 3×3 *magic square* is a 3×3 matrix with real entries, with the property that the sum of each row, each column, and each of the two main diagonals is the same.

Find three examples of 3×3 magic squares.
Show that the set of 3×3 magic squares forms a subspace of $\mathcal{M}_{3\times 3}(\mathbb{R})$. What is its dimension?