Linear Algebra I, Sheet 4, MT2019

Bases and subspaces. Dimension formula. Direct sums.

Main course

1. Find the row rank of each of the following matrices:

$$\text{(a)} \begin{pmatrix} 2 & 4 & -3 & 0 \\ 1 & -4 & 3 & 0 \\ 3 & -5 & 2 & 1 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 5 & 2 \end{pmatrix} \qquad \text{(c)} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 2 \\ 3 & 4 & 5 & 2 \end{pmatrix}$$

2. Let V be a vector space and let U, W be subspaces of V. Show that $U \setminus W$ is never a subspace. Find a necessary and sufficient condition for $U \cup W$ to be a subspace.

3. Let $V = \mathbb{R}^n$ where $n \geqslant 2$, and let U and W be subspaces of V of dimension $n-1$.

(a) Show that if $U \neq W$ then $\dim(U \cap W) = n-2$.

(b) Now suppose that $n \geqslant 3$ and let U_1, U_2, U_3 be three distinct subspaces of dimension $n-1$. Must it be true that $\dim(U_1 \cap U_2 \cap U_3) = n-3$? Give a proof or find a counterexample.

(c) Show that if $\dim U \leqslant n-2$ then there are infinitely many different subspaces X such that $U \leqslant X \leqslant V$.

4. Let $V = \mathbb{R}^4$, and let

$$X = \{(x_1, x_2, x_3, x_4) \in V : x_2 + x_3 + x_4 = 0\},$$
$$Y = \{(x_1, x_2, x_3, x_4) \in V : x_1 + x_2 = x_3 - 2x_4 = 0\}.$$

Find bases for X, Y, $X \cap Y$ and $X + Y$, and write down the dimensions of these subspaces.

5. Let V be an n-dimensional vector space.

(a) Prove that V contains a subspace of dimension r for each r in the range $0 \leqslant r \leqslant n$.

(b) Show that if $U_0 \lneqq U_1 \lneqq \cdots \lneqq U_k \leqslant V$ (strict containments of subspaces of V) then $k \leqslant n$. Show also that if $k = n$ then $\dim U_r = r$ for $0 \leqslant r \leqslant k$.

(c) Let U, W be subspaces of V such that $U \leqslant W$. Show that there is a subspace X of V such that $W \cap X = U$ and $W + X = V$.

6. (a) Show that $V = U \oplus W$ if and only if every vector $v \in V$ can be expressed uniquely in the form $v = u + w$ with $u \in U$ and $w \in W$.

(b) Let $V = \mathbb{R}^3$ and $U = \{(x_1, x_2, x_3) \in V : x_1 + x_2 + x_3 = 0\}$. For each of the following subspaces W, either prove that $V = U \oplus W$, or explain why this is not true:

(i) $W = \{(x, 0, -x) : x \in \mathbb{R}\}$;

(ii) $W = \{(x, 0, x) : x \in \mathbb{R}\}$;

(iii) $W = \{(x_1, x_2, x_3) \in V : x_1 = x_3\}$.

Starter

S1. For each of the following sets S in a vector space V, find a basis for Span(S).

(i) $S = \{(1, 0, 3), (-2, 5, 4)\} \subseteq \mathbb{R}^3$

(ii) $S = \{(6, 2, 0, -1), (3, 5, 9, -2), (-1, 0, 7, 8), (5, 5, -1, 2)\} \subseteq \mathbb{R}^4$

(iii) $S = \{(7, -3, 2, 11, 2), (0, 4, 9, -5, 16), (20, -13, 16, 24, 38), (1, 12, 8, -1, 0)\} \subseteq \mathbb{R}^5$

S2. Prove Proposition 26 from the notes: Let U, W be subspaces of a finite-dimensional vector space V. The following are equivalent:

(i) $V = U \oplus W$;

(ii) every $v \in V$ has a unique expression as $u + w$ where $u \in U$ and $w \in W$;

(iii) $\dim V = \dim U + \dim W$ and $V = U + W$;

(iv) $\dim V = \dim U + \dim W$ and $U \cap W = \{0_V\}$;

(v) if $u_1, \ldots, u_m$ is a basis for U and $w_1, \ldots, w_n$ is a basis for W, then $u_1, \ldots, u_m, w_1, \ldots, w_n$ is a basis for V.

[Note that proving the equivalence of (i) and (ii) is Q6(a) on this sheet!]

S3. Find an example to show that it is not the case that if $V = U \oplus W$ then every basis of V is a union of a basis of U and a basis of W.

Pudding

P1. Find the column rank of each of the matrices in Q1 on this sheet. What do you notice?

P2. Taking A as each of the three matrices in Q1 on this sheet, find all the solutions to $Ax = 0$ in each case. What do you notice?

P3. Let $V = V_1 \oplus V_2$. Take a subspace $U \leqslant V$. Must it be true that $U = U_1 \oplus U_2$ where $U_1 \leqslant V_1$ and $U_2 \leqslant V_2$? (Find a proof or give a counterexample.)