Linear Algebra I, Sheet 5, MT2019

Linear transformations. Kernel and image. Rank and nullity.

Main course

1. Which of the following formulae describe linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^3$?

(i) $T(x, y, z) = (y, z, 0)$;

(ii) $T(x, y, z) = (|x|, -z, 0)$;

(iii) $T(x, y, z) = (x - 1, x, y)$;

(iv) $T(x, y, z) = (yz, zx, xy)$.

As always, justify your answers.

2. Describe the kernel and image of each of the following linear transformations, and in each case find the nullity and the rank.

(i) $T : \mathbb{R}^4_{\text{col}} \to \mathbb{R}^3_{\text{col}}$ given by $T(X) = AX$ for $X \in \mathbb{R}^4_{\text{col}}$, where $A = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 2 & -1 & 1 \\ 0 & 3 & -2 & 0 \end{pmatrix}$.

(ii) $V = \mathcal{M}_{n\times n}(\mathbb{R})$, and $T : V \to \mathbb{R}$ is given by $T(X) = \operatorname{tr}(X)$, the sum $x_{11} + x_{22} + \cdots + x_{nn}$ of the entries on the main diagonal of X.

3. Let $V = \mathbb{R}_n[x]$, the vector space of real polynomials of degree $\leqslant n$. Define $D : V \to V$ to be differentiation with respect to x. Find the rank and nullity of D.

4. Let V be a finite-dimensional vector space, and let $S, T : V \to V$ be linear transformations.

(i) Show that $\operatorname{Im}(S + T) \leqslant \operatorname{Im} S + \operatorname{Im} T$. Deduce that $\operatorname{rank}(S + T) \leqslant \operatorname{rank} S + \operatorname{rank} T$.

(ii) Show that $\operatorname{null}(ST) \leqslant \operatorname{null} S + \operatorname{null} T$. [Hint: Focus on the restriction of S to $\operatorname{Im} T$, and consider its image and kernel.]

5. Let V be a finite-dimensional vector space.

(a) Let U, W be subspaces such that $V = U \oplus W$. Let $P : V \to V$ be the projection onto U along W, and let $Q : V \to V$ be the projection onto W along U.

(i) Show that $Q = \operatorname{id}_V - P$.

(ii) Show that $P^2 = P$, that $Q^2 = Q$ and that $PQ = QP = 0$.

(b) Now let $T : V \to V$ be a linear transformation such that $T^2 = T$ (such linear transformations are said to be *idempotent*).

(i) For $v \in V$ let $u = Tv$ and let $w = v - Tv$. Show that $u \in \operatorname{Im} T$, $w \in \ker T$ and $v = u + w$.

(ii) Show that $\operatorname{Im} T \cap \ker T = \{0\}$.

(iii) Deduce that $V = U \oplus W$ where $U := \operatorname{Im} T$, $W := \ker T$, and that T is the projection onto U along W.

6. Let V be an n-dimensional vector space and let $T : V \to V$ be a linear transformation. Prove that the following statements are equivalent:

(a) $\operatorname{Im} T = \ker T$, and (b) $T^2 = 0$, n is even and $\operatorname{rank} T = \frac{1}{2}n$.

Starter

S1. Let V, W be vector spaces over $\mathbb{F}$, let $T : V \to W$. Prove that T is linear if and only if $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$ for all $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{F}$. *(This is part of Proposition 28 in the notes.)*

S2. Prove Proposition 29 from the notes: for vector spaces V and W over $\mathbb{F}$, the set of linear transformations $V \to W$ forms a vector space (with pointwise addition and scalar multiplication).

S3. For each of the following maps, show that it is linear, find the kernel and image, and find the rank and nullity. Check that the Rank-Nullity Theorem is satisfied in each case.

(i) $T_1 : \mathbb{R}^4 \to \mathbb{R}^3$ given by $T(x_1, x_2, x_3, x_4) = (x_2 + x_3 - 2x_4, x_1 - x_2, x_1 - x_3 + 4x_4)$

(ii) $T_2 : \mathbb{R}^3_{\text{col}} \to \mathbb{R}^2_{\text{col}}$ given by $T_2(x) = Ax$ where $A = \begin{pmatrix} 3 & 4 & 7 \\ -1 & 0 & -6 \end{pmatrix}$.

(iii) $T_3 : \mathcal{M}_{m\times n}(\mathbb{R}) \to \mathcal{M}_{n\times m}(\mathbb{R})$ given by $T_3(X) = X^T$.

Pudding

P1. Consider the usual dot product on $\mathbb{R}^3$: we define $(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = x_1y_1 + x_2y_2 + x_3y_3$. Fix a vector $(a_1, a_2, a_3) \in \mathbb{R}^3$, and define a map $T : \mathbb{R}^3 \to \mathbb{R}$ by $T(x_1, x_2, x_3) = (x_1, x_2, x_3)\cdot(a_1, a_2, a_3)$. Show that T is linear. Find its kernel and image, and its rank and nullity.

P2. Let $T : \mathbb{R}^7 \to \mathbb{R}^4$ be a linear map. What are the possible nullities of T? Justify your answer. For each such possibility, give an example of a linear map $T : \mathbb{R}^7 \to \mathbb{R}^4$ with that nullity.

P3. Consider the real vector space V consisting of all real (infinite) sequences. Consider the map $T : V \to V$ given by $T_1(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$. Show that T_1 is linear. What is the kernel of T_1? What is its image?

Define T_2 by $T_2(x_1, x_2, x_3, \dots) = (x_1 + x_2, 0, x_3, x_4, 0, 0, 0, \dots)$. Show that T_2 is linear. What is the kernel of T_2? What is its image?

Can you give an example of a linear map $T_3 : V \to V$ where neither the kernel nor the image of T_3 has finite dimension?