Linear Algebra I, Sheet 6, MT2019

Matrices of linear transformations, change of basis, rank

Main course

1. Let V be a finite-dimensional vector space, and let U, W be subspaces such that $V = U \oplus W$. Let $T : V \to V$ be a linear transformation with the property that $T(U) \subseteq U$ and $T(W) \subseteq W$.

(a) Show that the matrix of T with respect to a basis of V which is the union of bases of U and of W has block form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where A is the matrix of the restriction of T to U and B is the matrix of the restriction of T to W.

(b) Now let $V = \mathbb{R}^4$ and define $T(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_1 - x_2, x_4, 0)$. Find 2-dimensional subspaces U and W such that $T(U) \subseteq U$, $T(W) \subseteq W$, and $V = U \oplus W$.

(c) For the subspaces U, W you found in (b), find bases B_U for U and B_W for W, and find the matrix of T with respect to $B_U \cup B_W$ as in (a).

2. Let $A \in \mathcal{M}_{n\times n}(\mathbb{R})$. Show that if $A^2 = 0$ then $\operatorname{rank} A \leqslant \frac{1}{2}n$. Show more generally that if $A^k = 0$ then $\operatorname{rank} A \leqslant n(k-1)/k$.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map $T(x, y, z) = (y, -x, z)$ for all $(x, y, z) \in \mathbb{R}^3$. Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^3$, and let $\mathcal{F}$ be the basis f_1, f_2, f_3 where $f_1 = (1, 1, 1)$, $f_2 = (1, 1, 0)$ and $f_3 = (1, 0, 0)$.

(a) Calculate the matrix A of T with respect to the standard basis $\mathcal{E}$ of $\mathbb{R}^3$.

(b) Calculate (directly) the matrix B of T with respect to the basis $\mathcal{F}$.

(c) Let I be the identity map of $\mathbb{R}^3$. Calculate the matrix P of I with respect to the bases $\mathcal{E}$, $\mathcal{F}$ and the matrix Q of I with respect to the bases $\mathcal{F}$, $\mathcal{E}$, and check that $PQ = I_3$.

(d) What do you predict is true about QBP? Now compute it—were you right?

4. (a) Show that if $X \in \mathcal{M}_{m\times n}(\mathbb{R})$ and $Y \in \mathcal{M}_{n\times m}(\mathbb{R})$ then $\operatorname{tr}(XY) = \operatorname{tr}(YX)$.

(b) Deduce that if A and B are similar $n \times n$ matrices then $\operatorname{tr}(A) = \operatorname{tr}(B)$.

5. Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ and let $b \in \mathbb{R}^m_{\text{col}}$. Prove that

(a) if $m < n$ then the system of linear equations $Ax = 0$ always has a non-trivial solution;

(b) if $m < n$ then the system of linear equations $Ax = b$ has either no solution or infinitely many different solutions;

(c) if A has rank m then the system of linear equations $Ax = b$ always has a solution;

(d) if A has rank n then the system of linear equations $Ax = b$ has at most one solution;

(e) if $m = n$ and A has rank n, then the system $Ax = b$ has precisely one solution.

6. Let $n \geqslant 2$ and let $V_n := \{a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n : a_0, a_1, \ldots, a_n \in \mathbb{R}\}$, the real vector space of homogeneous polynomials of degree n in two variables x and y. Let B_n be the "natural" ordered basis x^n, $x^{n-1}y$, $\ldots$, xy^{n-1}, y^n of V_n. Define $L : V_n \to V_{n-2}$ by $L(f) := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$.

(a) Check that L (the two-variable Laplace operator) is a linear transformation.

(b) Find the matrix of L with respect to the bases B_n of V_n and B_{n-2} of V_{n-2}.

(c) Find the rank of this matrix, and hence find the dimension of $\{f \in V_n : L(f) = 0\}$.

Starter

S1. In each part below, you are given two vector spaces V and W, a linear map $T : V \to W$, and ordered bases B_V and B_W for V and W respectively. Find the matrix A of T with respect to these bases.

(i) $V = W = \mathbb{R}^3$, with standard basis $B_V = B_W$, and $T(x, y, z) = (y, z, 0)$.

(ii) $V = \mathbb{R}^5$ with standard basis B_V, and $W = \mathbb{R}^3$ with ordered basis $w_1 = (-1, 1, 0)$, $w_2 = (1, 0, 1)$, $w_3 = (0, 2, 1)$, and $T(x_1, x_2, x_3, x_4, x_5) = (x_1 - x_3 + x_4, x_1 + x_2, x_5)$.

(iii) $V = W = \mathbb{R}^3$, with ordered bases B_V given by $v_1 = (3, -2, 0)$, $v_2 = (1, 1, 1)$, $v_3 = (4, 7, -2)$ and B_W given by $w_1 = (1, 0, 0)$, $w_2 = (0, 1, 1)$, $w_3 = (0, 1, -1)$, and T the identity map.

S2. Let $V = \mathbb{R}_3[x]$ be the real vector space of polynomials of degree at most 3 (as in Sheet 5 Q3). Let B_3 be the ordered basis x^3, x^2, x, 1. For each of the following linear maps $T : V \to V$, find the matrix for T with respect to B_3.

(i) $T = D$, differentiation.

(ii) $T(p(x)) = p(x + 1)$.

(iii) $T(p(x)) = \int_0^1 (t - x)^3 p(t)dt$

S3. For each of the following matrices, use elementary row operations and elementary column operations to reduce the matrix to a block matrix with an identity matrix in the top left-hand corner and 0 elsewhere (as in Proposition 47 of the notes).

$$\text{(a)} \begin{pmatrix} 1 & 2 & -1 \\ 3 & 7 & 4 \\ 5 & -1 & 6 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 0 & 1 & 4 & 9 \\ -2 & 8 & -7 & 3 \\ 1 & 5 & -6 & 0 \end{pmatrix} \qquad \text{(c)} \begin{pmatrix} 7 & -1 & 2 \\ 1 & 0 & -3 \\ 5 & -1 & 8 \end{pmatrix}$$

Pudding

P1. S2 above asks you to find the matrix A for differentiation of polynomials of degree at most 3, with respect to the ordered basis x^3, x^2, x, 1. Now find the matrix B with respect to the ordered basis 1, x, x^2, x^3. Can you find a matrix P such that $B = P^{-1}AP$?

P2. Is it true that if A, B and C are $n \times n$ real matrices then $\mathrm{tr}(ABC) = \mathrm{tr}(BAC)$? Is it true that $\mathrm{tr}(ABC) = \mathrm{tr}(BCA)$? (Give a proof or counterexample for each.)

P3. Let A be an $n \times n$ real matrix. For each of the following, give a proof or a counterexample.

(i) If A is similar to 0, then $A = 0$.

(ii) If A is similar to I_n, then $A = I_n$.

(iii) If A is similar to a diagonal matrix, then A is diagonal.

(iv) If A is similar to a symmetric matrix, then A is symmetric.

(v) If A is similar to B, then A^2 is similar to B^2.