

Linear Algebra I, Sheet 7, MT2019

Bilinear forms and inner product spaces

Main course

1. Let V be a real vector space. A bilinear form $\langle -, - \rangle$ on V is said to be *skew-symmetric* if $\langle v_1, v_2 \rangle = -\langle v_2, v_1 \rangle$ for all $v_1, v_2 \in V$. It is said to be *alternating* if $\langle v, v \rangle = 0$ for all $v \in V$.

(a) Show that every bilinear form on V may be written uniquely as the sum of a symmetric bilinear form and a skew-symmetric bilinear form.

(b) Show that a bilinear form $\langle -, - \rangle$ on V is alternating if and only if it is skew-symmetric.

2. Let V be a real vector space and let $\langle -, - \rangle$, $\langle -, - \rangle_1$ and $\langle -, - \rangle_2$ be inner products on V . The *norm* of a vector v is defined by $\|v\| := \sqrt{\langle v, v \rangle}$. Norms $\|v\|_1$ and $\|v\|_2$ are defined analogously.

(a) Show that if $u, v \in V$ then $\langle u, v \rangle = \frac{1}{2}(\|u + v\|^2 - \|u\|^2 - \|v\|^2)$.

(b) Deduce that if $\|x\|_1 = \|x\|_2$ for all $x \in V$ then $\langle u, v \rangle_1 = \langle u, v \rangle_2$ for all $u, v \in V$.

3. (a) Let V be a 2-dimensional real vector space with basis $\{e_1, e_2\}$. Describe all the inner products $\langle -, - \rangle$ on V for which $\langle e_1, e_1 \rangle = 1$ and $\langle e_2, e_2 \rangle = 1$.

(b) Show that if V is the 3-dimensional real vector space $\mathbb{R}_2[x]$ of polynomials of degree at most 2 in x , then the definition $\langle f(x), g(x) \rangle := f(0)g(0) + f(1)g(1) + f(2)g(2)$ describes an inner product on V .

4. Let V be an n -dimensional real vector space with inner product $\langle -, - \rangle$, and let U be an m -dimensional subspace of V . Define $U^\perp := \{v \in V : \langle v, u \rangle = 0 \text{ for all } u \in U\}$.

(a) Show that U^\perp is a subspace of V , and that $U^\perp \cap U = \{0\}$.

(b) Let u_1, \dots, u_m be a basis for U . Define $T : V \rightarrow \mathbb{R}^m$ by $T(v) = (x_1, \dots, x_m)$, where $x_i := \langle v, u_i \rangle$ for $i = 1, \dots, m$. Show that T is a linear transformation.

(c) For T as in (b), show that $\ker T = U^\perp$ and that $\text{rank } T = m$.

(d) Deduce that $\dim U^\perp = n - m$ and that $V = U \oplus U^\perp$.

(e) Let $V := \mathbb{R}_2[x]$ with the inner product defined in Q3(b), and let U be the subspace spanned by 1 and x . Find polynomials $p(x) \in U$ and $q(x) \in U^\perp$ such that $x^2 = p(x) + q(x)$.

5. Let a, b, c be real numbers. Use the Cauchy-Schwarz inequality to show that if $x^2 + y^2 + z^2 = 1$ then $ax + by + cz \leq \sqrt{a^2 + b^2 + c^2}$. At what points does equality hold?

6. Let V be a complex vector space, and let $\langle -, - \rangle : V \times V \rightarrow \mathbb{C}$ be a *sesquilinear* form: that is, for all $u, v, w \in V$ and all $\alpha, \beta \in \mathbb{C}$,

$$(i) \quad \langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle \quad \text{and} \quad (ii) \quad \langle u, v \rangle = \overline{\langle v, u \rangle}.$$

Note that $\langle v, v \rangle \in \mathbb{R}$ for all $v \in V$. If $\langle v, v \rangle > 0$ for all $v \in V \setminus \{0\}$ then the form is said to be *Hermitian*.

(a) Show that $\langle u, \alpha v + \beta w \rangle = \bar{\alpha} \langle u, v \rangle + \bar{\beta} \langle u, w \rangle$ for all $u, v, w \in V$ and all $\alpha, \beta \in \mathbb{C}$.

(b) Show that if $\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = x_1 \bar{y}_1 + \dots + x_n \bar{y}_n$ then $\langle -, - \rangle$ is a Hermitian form on \mathbb{C}^n .

(c) Now suppose that V is finite-dimensional over \mathbb{C} and that the form $\langle -, - \rangle$ is Hermitian. Let U be a subspace of V , and, as in Q4, define $U^\perp := \{v \in V : \langle v, u \rangle = 0 \text{ for all } u \in U\}$. Show that $U^\perp \leq V$ and that $V = U \oplus U^\perp$.

Starter

S1. Let $V = \mathcal{M}_{2 \times 2}(\mathbb{R})$. For $A, B \in \mathcal{M}_{2 \times 2}(\mathbb{R})$, define $\langle A, B \rangle = \text{tr}(A^T B)$. Does this define an inner product on V ?

S2. For each given inner product space V and vector $u \in V$, find a basis for the space u^\perp .

(i) $V = \mathbb{R}^3$ with the usual dot product, $u = (3, -1, 2)$.

(ii) $V = \mathbb{R}_2[x]$ with inner product as in Q3(b) overleaf, $u(x) = x^2 - 5x + 6$.

S3. For $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \mathbb{C}^n$, define $\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = x_1 y_1 + \dots + x_n y_n$. Does this define a Hermitian form on \mathbb{C}^n ? (Compare with Q6(b) overleaf.)

Pudding

P1. Let V be a real inner product space. Suppose that $v_1, v_2 \in V$ have the property that $\langle v_1, v \rangle = \langle v_2, v \rangle$ for all $v \in V$. Does this mean that $v_1 = v_2$? Give a proof or counterexample.

P2. Let $V = \mathbb{R}_2[x]$ with the inner product defined in Q3(b) overleaf. Find an orthonormal basis of V .

P3. Let V be a real vector space with inner product $\langle -, - \rangle$ and associated length function $\| \cdot \|$. Show that $\|v_1 + v_2\|^2 + \|v_1 - v_2\|^2 = 2\|v_1\|^2 + 2\|v_2\|^2$ for all $v_1, v_2 \in V$ — this is called the *parallelogram law*.