Linear Algebra I, Sheet 1, MT2019

Starter

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

Vicky Neale (vicky.neale@maths)

S1. Let $A := \begin{pmatrix} -1 & 3 \\ 1 & 2 \\ 7 & -2 \end{pmatrix}$, $B := \begin{pmatrix} 4 & -3 & 1 \\ 3 & -2 & 1 \\ 7 & 0 & 1 \end{pmatrix}$, $C := \begin{pmatrix} 2 & -2 & 3 \\ 0 & 0 & 0 \\ 5 & -4 & 3 \end{pmatrix}$. For which pairs X, $Y \in \{A, B, C\}$ is XY defined? When it is defined, calculate it.

In order for XY to be defined, we need the number of columns of X to be equal to the number of rows of Y. Here, this means that we can calculate B^2, C^2, BC, CB, BA and CA, but not A^2, AB or AC.

I don't remember "the number of columns of the first matrix must equal the number of rows of the second", because I'd have a high chance of misremembering that. Instead, I remember how matrix multiplication works, and figure it out from that. If you're not sure what I mean, try working out AB and see what goes wrong.

When we do the calculations, we find

$$B^2 = \begin{pmatrix} 4 & -3 & 1 \\ 3 & -2 & 1 \\ 7 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & -3 & 1 \\ 3 & -2 & 1 \\ 7 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 14 & -6 & 2 \\ 13 & -5 & 2 \\ 35 & -21 & 8 \end{pmatrix}$$

$$C^2 = \begin{pmatrix} 2 & -2 & 3 \\ 0 & 0 & 0 \\ 5 & -4 & 3 \end{pmatrix} \begin{pmatrix} 2 & -2 & 3 \\ 0 & 0 & 0 \\ 5 & -4 & 3 \end{pmatrix} = \begin{pmatrix} 19 & -16 & 15 \\ 0 & 0 & 0 \\ 25 & -22 & 24 \end{pmatrix}$$

$$BC = \begin{pmatrix} 4 & -3 & 1 \\ 3 & -2 & 1 \\ 7 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 & 3 \\ 0 & 0 & 0 \\ 5 & -4 & 3 \end{pmatrix} = \begin{pmatrix} 13 & -12 & 15 \\ 11 & -10 & 12 \\ 19 & -18 & 24 \end{pmatrix}$$

$$CB = \begin{pmatrix} 2 & -2 & 3 \\ 0 & 0 & 0 \\ 5 & -4 & 3 \end{pmatrix} \begin{pmatrix} 4 & -3 & 1 \\ 3 & -2 & 1 \\ 7 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 23 & -2 & 3 \\ 0 & 0 & 0 \\ 29 & -7 & 4 \end{pmatrix}$$

$$BA = \begin{pmatrix} 4 & -3 & 1 \\ 3 & -2 & 1 \\ 7 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 1 & 2 \\ 7 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 2 & 3 \\ 0 & 19 \end{pmatrix}$$

$$CA = \begin{pmatrix} 2 & -2 & 3 \\ 0 & 0 & 0 \\ 5 & -4 & 3 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ 1 & 2 \\ 7 & -2 \end{pmatrix} = \begin{pmatrix} 17 & -4 \\ 0 & 0 \\ 12 & 1 \end{pmatrix}.$$

I noticed a couple of things when I did these calculations.

When the matrix on the left had a row of 0s, it was really easy to do that multiplication. That wasn't true when the matrix on the right had a row of 0s.

When the matrix on the right had a column of 1s, it was easy to do that multiplication (add the entries). That wasn't true when the matrix on the left had a column of 1s.

I invite you to think about what this tells us about matrix multiplication.

S2. Calculate the following two matrix products.

$$\begin{pmatrix} x & y & z & w \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}; \quad \begin{pmatrix} x & y & z & w \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

For the first one: the matrix in the middle is a 4×4 identity matrix, which means that multiplying by it has no effect. So we find

$$\begin{pmatrix} x & y & z & w \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} x & y & z & w \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = x^2 + y^2 + z^2 + w^2.$$

When we multiply out the second one, we get $x^2 - y^2 - z^2 - w^2$.

This second one turns out to be important in Special Relativity. You'll learn more about it if you choose the Special Relativity short option in Part A (`https://courses.maths.ox.ac.uk/node/44178`).

S3. Prove Proposition 5 from the notes:
Let A, B be invertible $n \times n$ matrices. Then AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

Since A and B are invertible, we can consider their inverses, A^{-1} and B^{-1} respectively.

From the definition of the inverse, we know that this means that $AA^{-1} = I_n = A^{-1}A$ and $BB^{-1} = I_n = B^{-1}B$.

Now

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$$

and also

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n,$$

so AB is invertible and its inverse is $B^{-1}A^{-1}$.