Linear Algebra I, Sheet 1, MT2019

Pudding

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

I'm not going to give full details/proofs for every question, but hopefully I'll give something useful against which you can compare your thinking.

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P1.

(a) Is every diagonal matrix invertible? What can you say about the inverse of an invertible diagonal matrix?

(b) Is every upper triangular matrix invertible? What can you say about the inverse of an invertible upper triangular matrix?

(a) Not every diagonal matrix is invertible.

An extreme example is a zero matrix (all entries 0), which is diagonal but not invertible.

More generally, a diagonal matrix is invertible if and only if all its diagonal entries are nonzero. (See also problem S1 on this sheet, and think about what happens when you multiply a matrix with a row of 0s by another matrix.)

Let $A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$ be an invertible matrix, so that $a_1, \ldots, a_n$ are nonzero. Then

$$A^{-1} = \begin{pmatrix} a_1^{-1} & 0 & \cdots & 0 \\ 0 & a_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^{-1} \end{pmatrix}.$$

(b) Not every upper triangular matrix is invertible.

The zero matrix gives a counterexample here too.

If A is an upper triangular matrix, then its inverse is also upper triangular. The diagonal entries behave similarly to the case of a diagonal matrix, but the entries above the diagonal are more complicated.

P2.

(a) What happens if we multiply an upper triangular matrix by an upper triangular matrix?

(b) What if we multiply a lower triangular matrix by a lower triangular matrix?

(c) What if we multiply a lower triangular matrix by an upper triangular matrix?

I don't expect you to look at this and immediately know the answer. I might well start a question like this by playing with examples, even though this is not what the question asks for. I'd test with small matrices first, say 2×2 and 3×3 matrices, to get a feel for what's going on. I might well experiment with numbers, before moving to algebra, but different people have different tastes. But don't be afraid to play around with examples to help you get a feel for what's going on, you're not expected to look at questions and immediately know what to do.

(a) The product of two upper triangular matrices is again upper triangular.

(b) The product of two lower triangular matrices is again lower triangular.

(c) We can't say much about the form of the product of a lower triangular matrix and an upper triangular matrix: we can get matrices of lots of shapes in this way.

But flipping the question around turns out to be interesting. Instead of asking "Take a lower triangular matrix L and an upper triangular matrix U, what can we say about LU?", we can instead ask "Here's a matrix, can we factorise it in the form LU?". This is an important question, and is explored in the Part A Numerical Analysis course `https://courses.maths.ox.ac.uk/node/44065`.

P3.

(a) Let A and B be orthogonal $n \times n$ matrices. Must their product AB also be orthogonal?

(b) Give an example of a 2×2 orthogonal matrix. Give another example. And another. And another. And another. And another. Can you find all possible 2×2 orthogonal matrices?

(a) Yes, the product of two orthogonal matrices is orthogonal. If A and B are orthogonal, then by definition $AA^T = I_n = A^T A$ and $BB^T = I_n = B^T B$.

Then, using properties of the transpose as in Q5 on Sheet 1, $(AB)(AB)^T = (AB)(B^T A^T) = A(BB^T)A^T = AI_n A^T = AA^T = I_n$ and similarly $(AB)^T(AB) = I_n$, so that AB is orthogonal.

You might like to compare with my solution to S3 on this sheet. Why are the arguments so similar?

This property of orthogonal matrices, that they are "closed under multiplication", is important. In particular, it is one of the properties of orthogonal matrices that makes the set of $n \times n$ orthogonal matrices into a group under multiplication. See the Prelims Groups and Group Actions course `https://courses.maths.ox.ac.uk/node/43835`.

(b) Here are some examples of 2×2 orthogonal matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

I asked for five examples so that you'd have to go beyond just using 0, 1 and -1 as entries!

Orthogonal matrices are important for their geometrical significance. If you are doing Geometry this term, you'll learn more about this. Whether or not you are doing the Geometry course, you can find a detailed solution to finding all 2×2 orthogonal matrices in Example 67 of Richard Earl's lecture notes for the Geometry course `https://courses.maths.ox.ac.uk/node/view_material/43919`.