Linear Algebra I, Sheet 2, MT2019

Starter

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

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S1. Use EROs to determine whether the following matrices are invertible. For each invertible matrix, find the inverse.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}; \qquad \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 2 & 4 & 6 \end{pmatrix}; \qquad \begin{pmatrix} -2 & 0 & 4 & 3 \\ 1 & 7 & 5 & -6 \\ -3 & 7 & 13 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}.$$

(a) For the first one, we have

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right) \xrightarrow[R_2 \to R_2 - 3R_1]{} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right) \xrightarrow[R_2 \to -\frac{1}{2}R_2]{} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right)$$

$$\xrightarrow[R_1 \to R_1 - 2R_2]{} \left(\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right).$$

This shows that we can reduce the original matrix to the identity matrix using a sequence of EROs, and so the original matrix is invertible.

Also, we can read off the inverse by seeing what effect the same sequence of EROs has on the identity matrix. We see that the matrix has inverse

$$\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

At this point I tend to check that I have got the arithmetic right, by multiplying the supposed inverse by the original matrix and checking that I do indeed get the identity matrix!

(b) This time, we have

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 4 & 6 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 & 2 \end{array}\right).$$

Now that the last row consists of 0s (on the left-hand side corresponding to the original matrix), we see that this matrix is not invertible, and we can stop here.

I did two operations at once here. Importantly, they were 'independent' of each other, so it doesn't matter which order I do them in. If I'd wanted to multiply row 1 by a scalar and then add a multiple of row 1 to row 2, then I would have done these in separate steps to avoid any confusion.

(c) Finally, we have

$$\left(\begin{array}{cccc|cccc} -2 & 0 & 4 & 3 & 1 & 0 & 0 & 0 \\ 1 & 7 & 5 & -6 & 0 & 1 & 0 & 0 \\ -3 & 7 & 13 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_1 \to -\frac{1}{2}R_1]{} \left(\begin{array}{cccc|cccc} 1 & 0 & -2 & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 1 & 7 & 5 & -6 & 0 & 1 & 0 & 0 \\ -3 & 7 & 13 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 + 3R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{cccc|cccc} 1 & 0 & -2 & -\frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 7 & 7 & -\frac{9}{2} & \frac{1}{2} & 1 & 0 & 0 \\ 0 & 7 & 7 & -\frac{9}{2} & -\frac{3}{2} & 0 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 1 \end{array}\right).$$

Now the second and third rows (on the left-hand side) are the same. This means that we can subtract one from the other, to get a matrix with a zero row. So the original matrix is not invertible.

I could continue to work out the matrix in reduced row echelon form, but this is not necessary because we can see already that the matrix is not invertible.

See question P2 for more on these matrices and another way to explore invertibility.

S2. Show that the inverse of an ERO is an ERO. (Hint: consider each of the three types of ERO separately.)

- One type of ERO is to swap two rows, say r and s. This is *self-inverse*: it is its own inverse, we simply swap them again to undo the operation.
- A second type of ERO is to multiply a row, say row r, by a nonzero scalar λ. To undo this, we can multiply the same row r by the nonzero scalar $\frac{1}{\lambda}$, so the inverse is again an ERO, and in fact is of the same type.
- The final type of ERO is to add λ times row r to row s (where $r \neq s$). To undo this, we can add $-\lambda$ times row r to row s, so the inverse is again an ERO, of the same type.

S3. Prove Lemma 9 from the lecture notes: let V be a vector space over $\mathbb{F}$. Then there is a unique additive identity element 0_V.

Suppose that 0_V and $0'_V$ are both additive identity elements in V. *[Secret aim: $0_V = 0'_V$.]*

Then, by definition, $v + 0_V = v = 0_V + v$ for all $v \in V$, and similarly $v + 0'_V = v = 0'_V + v$ for all $v \in V$.

Now $0_V = 0_V + 0'_V = 0'_V$, where the first equality follows because $0'_V$ is an additive identity and the second because 0_V is an additive identity.