Linear Algebra I, Sheet 2, MT2019
Pudding

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

I'm not going to give full details/proofs for every question, but hopefully I'll give something useful against which you can compare your thinking.

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P1. Use EROs to explore for which real numbers a, b, c, d the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible.

The key to this question is to keep careful track of the various cases—we must be sure that we aren't dividing by 0, but also that we have covered all eventualities.

Case 1: $a \neq 0$.

We have

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \xrightarrow[R_1 \to \frac{1}{a} R_1]{} \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right) \xrightarrow[R_2 \to R_2 - \frac{c}{R_1}]{} \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d - \frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right).$$

If $d - \frac{bc}{a} = 0$, then we have a zero row and so the matrix is not invertible.

So we continue on the assumption that $d - \frac{bc}{a} \neq 0$. Applying further EROs, we get

$$\xrightarrow[R_2 \to \frac{a}{ad-bc} R_2]{} \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right) \xrightarrow[R_1 \to R_1 - \frac{b}{a} R_2]{} \left(\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right).$$

We conclude that if $d - \frac{bc}{a} \neq 0$, then the matrix is invertible, with inverse

$$\frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Case 2: $a = 0$.

We use EROs again:

$$\left(\begin{array}{cc|cc} 0 & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \xrightarrow[R_1 \leftrightarrow R_2]{} \left(\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{array}\right).$$

If $b = 0$, then the matrix is not invertible.

We proceed assuming that $b \neq 0$, and continue to apply EROs:

$$\xrightarrow[R_2 \to \frac{1}{b} R_2]{} \left(\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & 1 & \frac{1}{b} & 0 \end{array}\right) \xrightarrow[R_1 \to R_1 - dR_2]{} \left(\begin{array}{cc|cc} c & 0 & -\frac{d}{b} & 1 \\ 0 & 1 & \frac{1}{b} & 0 \end{array}\right).$$

If $c = 0$, then the matrix is not invertible.

Continuing on the assumption that $c \neq 0$, we get

$$\xrightarrow[R_1 \to \frac{1}{c} R_1]{} \left(\begin{array}{cc|cc} 1 & 0 & -\frac{d}{bc} & \frac{1}{c} \\ 0 & 1 & \frac{1}{b} & 0 \end{array}\right).$$

We see that if $a = 0$ and $bc \neq 0$ then the matrix is invertible, with inverse

$$\frac{1}{bc}\begin{pmatrix} -d & b \\ c & 0 \end{pmatrix}.$$

We seem to have different answers for the two cases (depending on whether $a = 0$ or not). Happily, when we look more closely at these we find that the answers are the same in both cases.

Having done our exploration, we might write up the argument by starting with a statement of the result and then giving a proof. For example, our statement might be

Claim The matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible if and only if $ad - bc \neq 0$. In the case that it is invertible, the inverse is

$$\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

I have chosen not to do that in these solutions, to illustrate how I might go about finding such a statement without already knowing the answer.

The quantity $ad - bc$ is called the *determinant* of the matrix. You will study determinants further in Linear Algebra II next term.

P2. Pick a matrix from question S1. Can you find any relationships between the rows of the matrix? Or between the columns? Does this relate to whether the matrix is invertible? Try this for each of the matrices from S1.

It would be a good idea to try question S1 before reading this solution, or at least to look at the solutions to that question.

Let's look at the second and third ones first.

We found using EROs that the second matrix is not invertible. Looking carefully at the EROs used, the key point was that $R_3 = 2R_1$, so we can apply an ERO to create a zero row.

More generally, we can see that if one row is a scalar multiple of another, then the matrix is not invertible because we can apply an ERO to create a zero row (or we already have a zero row).

In the third case, looking back at the EROs we see that the key was that $R_3 - \frac{3}{2}R_1 = R_2 + \frac{1}{2}R_1$. Rearranging, this becomes $2R_1 + R_2 - R_3 = 0$. This is a more complex relationship than one row being a scalar multiple of another, but could be thought of as a generalisation of it.

The name for this sort of relationship is that the rows R_1, R_2 and R_3 are *linearly dependent*, because there is a nontrivial *linear combination* of them that is 0. We'll explore these ideas more during the Linear Algebra I course, and you'll encounter them again in the context of invertibility of matrices when studying determinants in Linear Algebra II next term.

Examining the rows of the first matrix in S1, we see that there is no linear dependence between them: they are linearly independent. If $\lambda_1 R_1 + \lambda_2 R_2 = 0$, then $\lambda_1 = \lambda_2 = 0$.

It is possible to do something similar with the columns too. The columns of the first matrix in S1 are linearly independent. For the second matrix, we have $C_1 - 2C_2 + C_3 = 0$. For the third, we have $-81C_1 + 60C_2 - 51C_3 + 14C_4 = 0$.

The coefficients in the last of those are not the sorts of numbers that I can just spot by staring at the columns! To find them, I solved the equation

$$\begin{pmatrix} -2 & 0 & 4 & 3 \\ 1 & 7 & 5 & -6 \\ -3 & 7 & 13 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

using EROs to reduce the equation to a convenient form from which I could read off a solution. Can you see why this matrix equation is relevant? The fact that the matrix isn't invertible means that there isn't a unique choice for the coefficients, which is how we can hope to find nonzero coefficients for a linear dependence.

It might now be interesting to look back at P1, thinking about the invertibility of a general 2×2 matrix by considering whether the rows are linearly dependent.

P3. The two-dimensional plane $\mathbb{R}^2$ is a vector space. Which subsets of $\mathbb{R}^2$ themselves have the structure of a vector space (using the same operations of addition and scalar multiplication as in $\mathbb{R}^2$)?

This question is about investigating the *subspaces* of $\mathbb{R}^2$—we'll look at the notion of subspace in lectures.

The subspace $\{(0, 0)\}$ is not very exciting, but is a perfectly good vector space.

If we have a subspace containing a nonzero point, then we must have all scalar multiples of that point, so we have the whole of a straight line automatically (the straight line passing through the origin and the given point).

In fact, any straight line through the origin is a subspace of $\mathbb{R}^2$. (But a straight line not through the origin cannot be a subspace—can you see why?)

If we have a straight line and another point not on the line, then a consequence of the vector space axioms is that we must have the whole of $\mathbb{R}^2$. Can you prove this?

This relates to ideas of linear dependence and dimension, which we'll explore later in Linear Algebra I.