Linear Algebra I, Sheet 3, MT2019

Starter

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

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S1. Prove Proposition 14 from the lecture notes. That is, let V be a vector space, and take subspaces $U, W \leqslant V$. Then prove that $U + W \leqslant V$ and $U \cap W \leqslant V$.

Claim Let $U, W \leqslant V$. Then $U + W \leqslant V$.
Proof We use the Subspace Test.

- Since $U \leqslant V$, we have $0_V \in U$. Similarly, $W \leqslant V$ so $0_V \in W$.

  Now $0_V = 0_V + 0_V \in U + W$.

- Take $u_1 + w_1, u_2 + w_2 \in U + W$, where $u_1, u_2 \in U$ and $w_1, w_2 \in W$, and take $\lambda \in \mathbb{F}$ (where V is a vector space over the field $\mathbb{F}$).

  Then $\lambda(u_1 + w_1) + (u_2 + w_2) = (\lambda u_1 + u_2) + (\lambda w_1 + w_2) \in U + W$, since U and W are both subspaces.

So, by the Subspace Test, $U + W \leqslant V$. □

Claim Let $U, W \leqslant V$. Then $U \cap W \leqslant V$.
Proof We use the Subspace Test.

- Since $U \leqslant V$, we have $0_V \in U$. Similarly, $W \leqslant V$ so $0_V \in W$.

  Now $0_V \in U \cap W$.

- Take $v_1, v_2 \in U \cap W$, and take $\lambda \in \mathbb{F}$.

  Then $v_1, v_2 \in U$ and $v_1, v_2 \in W$.

  Then $\lambda v_1 + v_2 \in U$, since U is a subspace, and similarly $\lambda v_1 + v_2 \in W$.

  So $\lambda v_1 + v_2 \in U \cap W$.

So, by the Subspace Test, $U \cap W \leqslant V$. □

S2. For each of the following, give an example or prove that no such example exists, first when $V = \mathbb{R}^3$, and second when $V = \mathcal{M}_{2\times 2}(\mathbb{R})$.

(i) A set of 2 linearly independent vectors in V.

(ii) A set of 3 linearly independent vectors in V.

(iii) A set of 4 linearly independent vectors in V.

(iv) A spanning set of 2 vectors in V.

(v) A spanning set of 3 vectors in V.

(vi) A spanning set of 4 vectors in V.

For the parts where there are examples, there are many possible examples!

(a) Let $V = \mathbb{R}^3$.

(i) For example, $\{(1,0,0),(0,1,0)\}$ is linearly independent.
(But we cannot simply take any set of 2 vectors in V. For example, $\{(1,0,0),(2,0,0)\}$ is not linearly independent.)

(ii) For example, $\{(1,0,0),(0,1,0),(0,0,1)\}$ is linearly independent.

(iii) There are no sets of 4 linearly independent vectors in V. One way to see this is that V has dimension 3 — for example we see that the set in (ii) above is a basis with 3 elements — and so any linearly independent set has size at most 3.
We could also see it directly, rather than by quoting a result. Suppose that $a_1 = (a_{11}, a_{21}, a_{31})$, $a_2 = (a_{12}, a_{22}, a_{32})$, $a_3 = (a_{13}, a_{23}, a_{33})$, $a_4 = (a_{14}, a_{24}, a_{34})$ are four vectors in V.
[Secret aim: these four vectors are linearly dependent.]
Take λ_1, λ_2, λ_3, $\lambda_4 \in \mathbb{R}$ such that $\lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 = 0$.
[Secret aim: there is a solution with λ_1, λ_2, λ_3, λ_4 not all 0.]
Then

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

If we were to apply EROs to reduce the 3×4 matrix to RRE form, we would find that there is at least one free variable λ_i, because there can be at most three columns containing the leading entry of a row. But we can choose any value for a free variable, and so there is certainly a solution with λ_1, λ_2, λ_3, λ_4 not all 0.

(iv) There are no spanning sets of 2 vectors in V. Since V has dimension 3, any spanning set must contain at least 3 elements. (Or, again, we could see it directly.)

(v) For example, $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a spanning set.

(vi) For example, $\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$ is a spanning set.
(If we have a spanning set with 3 vectors, then we can add any fourth vector at all and still have a spanning set.)

(b) Let $V = \mathcal{M}_{2\times 2}(\mathbb{R})$.

(i) For example, $\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\}$ is linearly independent.

(ii) For example, $\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$ is linearly independent.

(iii) For example, $\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$ is linearly independent.

(iv) There are no spanning sets of 2 vectors in V. The linearly independent set in (iii) above is in fact the standard basis for V, which has dimension 4, so any spanning set must contain at least 4 elements.

(v) Similarly to (iv), there are no spanning sets of 3 vectors in V.

(vi) The example from (iii) is also a spanning set with 4 elements — it is a basis.

S3. Let V be the set of polynomials of degree at most 2 with real coefficients. That is, $V = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$. Show that this is a vector space (under the usual polynomial addition and scalar multiplication).
Give a basis B_1 for V. Give another basis B_2 that shares exactly one element with B_1. Give a third basis B_3 that shares no elements with B_1 or B_2.

To show that V is a vector space, we check the usual list of axioms, noting that V is indeed closed under addition and scalar multiplication (for example, adding two polynomials of degree at most 2 does give a polynomial of degree at most 2).

I'm not going to write out the whole list of axiom checks here!

There are many possible bases for V — this is the point of the question.

The standard basis, which I'm going to call B_1, is $B_1 = \{1, x, x^2\}$. We can see immediately that this is linearly independent and that it spans V.

An example of another basis that shares exactly one element with B_1 is $B_2 = \{1, x + x^2, x - x^2\}$. To show that this is a basis, since we already know that V has dimension 3 and this is a set of 3 elements, it is enough to prove either that B_2 spans or that B_2 is linearly independent (we don't need to prove both).

To show that B_2 is linearly independent: take λ_1, λ_2, $\lambda_3 \in \mathbb{R}$ such that $\lambda \cdot 1 + \lambda_2(x + x^2) + \lambda_3(x - x^2) = 0$. Comparing constant coefficients shows $\lambda_1 = 0$. Comparing coefficients of x and x^2 gives $\lambda_2 + \lambda_3 = 0$ and $\lambda_2 - \lambda_3 = 0$ respectively, and solving these simultaneous equations gives $\lambda_2 = \lambda_3 = 0$.

A third basis, sharing no elements with B_1 or B_2, is $B_3 = \{3 - x^2, \pi x, 1 + x + x^2\}$. Again, since this is a set with $\dim V$ elements, to prove that B_3 is a basis it suffices to prove that B_3 is linearly independent. We can do this using a similar argument to that used for B_2.