Linear Algebra I, Sheet 3, MT2019

Pudding

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

I'm not going to give full details/proofs for every question, but hopefully I'll give something useful against which you can compare your thinking.

Vicky Neale (vicky.neale@maths)

P1. Consider V the vector space of all real sequences. For $k \geqslant 1$, let $e^{(k)}$ be the sequence where all terms are 0 except for a 1 in position k. (So $e^{(1)} = (1, 0, 0, \ldots)$ and $e^{(2)} = (0, 1, 0, \ldots)$ for example.) Let $S = \{e^{(k)} : k \geqslant 1\}$. Is S linearly independent in V? Does S span V?

Note that a linear combination always involves only finitely many terms.

We see that S is linearly independent in V.

Since S is an infinite set, we need to show that every finite subset of S is linearly independent.

But if $\lambda_1 e^{(k_1)} + \cdots + \lambda_r e^{(k_r)} = 0$, then we must have $\lambda_1 = \ldots = \lambda_r = 0$ by looking at the entries in positions $k_1, \ldots, k_r$.

So S is indeed linearly independent.

However, S does not span V. For example, V contains a sequence $(1, 1, 1, \ldots)$ in which every term is 1. This sequence is not a linear combination of elements in S, because a linear combination involves only finitely many terms.

The question of whether this vector space has a basis is beyond the scope of this course. The current Part B course on Set Theory includes a proof that if we assume the Axiom of Choice then every vector space has a basis. This argument proves that a basis exists, but does not exhibit a concrete example of such a basis!

P2. Let $V = \mathbb{R}^4$. Let $W = \{(x_1, x_2, x_3, x_4) \in V : x_1 + 2x_2 - x_3 = 0\}$. Show that W is a subspace of V.
What is the dimension of W? Find a basis B_W of W.
Consider the standard basis B_V of V. Is there a subset of B_V that is a basis for W?
Can you add one or more vectors to your basis B_W for W to obtain a basis for V?
Can you generalise?

Claim $W \leqslant V$.
Proof We use the Subspace Test.

Certainly $(0, 0, 0, 0) \in W$.

Take (x_1, x_2, x_3, x_4), $(y_1, y_2, y_3, y_4) \in W$ and $\lambda \in \mathbb{R}$. Then $x_1 + 2x_2 - x_3 = 0$ and $y_1 + 2y_2 - y_3 = 0$.

Now $\lambda(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) = (\lambda x_1 + y_1, \lambda x_2 + y_2, \lambda x_3 + y_3, \lambda x_4 + y_4)$, and $(\lambda x_1 + y_1) + 2(\lambda x_2 + y_2) - (\lambda x_3 + y_3) = \lambda(x_1 + 2x_2 - x_3) + (y_1 + 2y_2 - y_3) = 0$, so $\lambda(x_1, x_2, x_3, x_4) + (y_1, y_2, y_3, y_4) \in W$.

So, by the Subspace Test, $W \leqslant V$. $\square$

Let $B_W = \{(1, 0, 1, 0), (0, 1, 2, 0), (0, 0, 0, 1)\}$. Note that all the elements of B_W are in W. A quick check shows that B_W is a linearly independent set. Also, if $(x_1, x_2, x_3, x_4) \in W$ then $x_3 = x_1 + 2x_2$ so $(x_1, x_2, x_3, x_4) = x_1(1, 0, 1, 0) + x_2(0, 1, 2, 0) + x_4(0, 0, 0, 1)$ is in the span of B_W. So B_W is a basis for W, and W has dimension 3.

The standard basis of V is $B_V = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)\}$. The first three of these vectors do not lie in W, so cannot be included in a basis of W. But a basis of W must contain three elements. So there is no subset of B_V that is a basis for W.

We can, however, add a vector to B_W to get a basis for V. We know that V has dimension 4, so any linearly independent set of 4 elements in V will be a basis. For example, $B_W \cup \{(0, 0, 1, 0)\} = \{(1, 0, 1, 0), (0, 1, 2, 0), (0, 0, 0, 1), (0, 0, 1, 0\}$ is a basis. (Quick check of linear independence: if $\lambda_1(1, 0, 1, 0) + \lambda_2(0, 1, 2, 0) + \lambda_3(0, 0, 0, 1) + \lambda_4(0, 0, 1, 0) = (0, 0, 0, 0)$ then, looking at each coordinate in turn $\lambda_1 = 0$ and $\lambda_2 = 0$ and $\lambda_1 + 2\lambda_2 + \lambda_4 = 0$ and $\lambda_3 = 0$, so $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$.)

We'll see in lectures that if we have a linearly independent set S in a finite-dimensional vector space V, then it is possible to extend S to a basis of V. This result is *extremely* helpful! But, as we see in P2, it is not in general possible to take a basis of V and to pick out a subset that gives a basis of a given subspace.

P3. A 3×3 *magic square* is a 3×3 matrix with real entries, with the property that the sum of each row, each column, and each of the two main diagonals is the same.
Find three examples of 3×3 magic squares.
Show that the set of 3×3 magic squares forms a subspace of $\mathcal{M}_{3\times 3}(\mathbb{R})$. What is its dimension?

Here are some not very exciting examples of magic squares:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 59 & 59 & 59 \\ 59 & 59 & 59 \\ 59 & 59 & 59 \end{pmatrix}.$$

Here are some more interesting examples of magic squares:

$$\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{pmatrix} \qquad \begin{pmatrix} 2 & 11 & 2 \\ 5 & 5 & 5 \\ 8 & -1 & 8 \end{pmatrix}.$$

Let S denote the set of 3×3 magic squares. We can use the subspace test to check that S is a subspace of $\mathcal{M}_{3\times 3}(\mathbb{R})$ (I'm not including the details here).

Here are three linearly independent magic squares:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

It turns out that they span the space of magic squares (although showing this is slightly fiddly), and so the space is a 3-dimensional subspace of $\mathcal{M}_{3\times 3}(\mathbb{R})$.

Another way to explore the dimension is to record the constraints involved in having a magic square. We see that $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ is a magic square if and only if a certain system of 8 simultaneous equations is satisfied. This system can be represented as the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \\ i \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \\ \alpha \\ \alpha \\ \alpha \\ \alpha \\ \alpha \end{pmatrix},$$

where α is the sum of each row/column/diagonal. The first three rows tell us that the sum of each row must be α. Rows 4, 5, 6 tell us that the sum of each column must be α. And finally rows 7 and 8 tell us that the sum of each diagonal must be α.

We can then use EROs to reduce the corresponding augmented matrix to RRE form. If you are an ERO enthusiast, then you can do this by hand; otherwise a computer will be happy to help. The augmented matrix reduces to

$$\left(\begin{array}{ccccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{2}{3}\alpha \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{2}{3}\alpha \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -\frac{1}{3}\alpha \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -2 & -\frac{2}{3}\alpha \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{3}\alpha \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & \frac{4}{3}\alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \alpha \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

On the left, there are 7 determined variables and 2 free. We can also choose any real value of α, which gives an extra degree of freedom. So we see that the dimension of the subspace of magic squares is 3.