Linear Algebra I, Sheet 4, MT2019
Starter

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

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S1. For each of the following sets S in a vector space V, find a basis for $\mathrm{Span}(S)$.

(i) $S = \{(1,0,3),(-2,5,4)\} \subseteq \mathbb{R}^3$

(ii) $S = \{(6,2,0,-1),(3,5,9,-2),(-1,0,7,8),(5,5,-1,2)\} \subseteq \mathbb{R}^4$

(iii) $S = \{(7,-3,2,11,2),(0,4,9,-5,16),(20,-13,16,24,38),(1,12,8,-1,0)\} \subseteq \mathbb{R}^5$

One strategy is to create a matrix whose rows are the vectors of S. Then $\mathrm{Span}(S)$ is precisely the row space of this matrix, which we can find by applying EROs to reduce the matrix to echelon form. But by looking carefully at the situation, it might be possible to use another, quicker, strategy too.

(a) We have two vectors in S, and it is quick to see that they are linearly independent, so S is itself a basis for $\mathrm{Span}(S)$.

(b) It's harder to see by eye whether these vectors are linearly independent this time, so let's try creating a matrix whose rows are the vectors of S, and reduce the matrix to echelon form.

We start with the matrix

$$\begin{pmatrix} 6 & 2 & 0 & -1 \\ 3 & 5 & 9 & -2 \\ -1 & 0 & 7 & 8 \\ 5 & 5 & -1 & 2 \end{pmatrix}.$$

When I used EROs to reduce it to echelon form, I got

$$\begin{pmatrix} 1 & \frac{1}{3} & 0 & -\frac{1}{6} \\ 0 & 1 & \frac{9}{4} & -\frac{3}{8} \\ 0 & 0 & 1 & \frac{191}{150} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So a basis for $\mathrm{Span}(S)$ is $\{(1,\frac{1}{3},0,-\frac{1}{6}),(0,1,\frac{9}{4},-\frac{3}{8}),(0,0,1,\frac{191}{150}),(0,0,0,1)\}$. These span $\mathrm{Span}(S)$ because EROs don't change the row space, and we can immediately see that they are linearly independent by looking at the leading entries.

If we had continued with EROs to reduce the matrix to RRE form, then we would have found that the standard basis is a basis for $\mathrm{Span}(S)$ *in this case.*

(c) Let's try the matrix-and-EROs strategy again. We start with the matrix

$$\begin{pmatrix} 7 & -3 & 2 & 11 & 2 \\ 0 & 4 & 9 & -5 & 16 \\ 20 & -13 & 16 & 24 & 38 \\ 1 & 12 & 8 & -1 & 0 \end{pmatrix}.$$

When I used EROs to reduce it to echelon form, I got

$$\begin{pmatrix} 1 & -\frac{3}{7} & \frac{2}{7} & \frac{11}{7} & \frac{2}{7} \\ 0 & 1 & \frac{9}{4} & -\frac{5}{4} & 4 \\ 0 & 0 & 1 & -\frac{121}{189} & \frac{200}{81} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

so a basis for $\mathrm{Span}(S)$ is $\{(1, -\frac{3}{7}, \frac{2}{7}, \frac{11}{7}, \frac{2}{7}), (0, 1, \frac{9}{4}, -\frac{5}{4}, 4), (0, 0, 1, -\frac{121}{189}, \frac{200}{81})\}$. We know that these span $\mathrm{Span}(S)$ because EROs don't change the row space, and we can immediately see that they are linearly independent by looking at the leading entries.

We could, if we wanted to, make the basis look nicer (have all the entries integers) by multiplying through—this has no effect on whether the set is a basis. So we also see that $\{(7, -3, 2, 11, 7), (0, 4, 9, -5, 16), (0, 0, 567, -363, 1400)\}$ is a basis.

[In fact, when I chose the vectors, I picked the third to be a linear combination of the others. The first, second and fourth are also linearly independent, so another answer is that $\{(7, -3, 2, 11, 2), (0, 4, 9, -5, 16), (1, 12, 8, -1, 0)\}$ *is also a basis of* $\mathrm{Span}(S)$*.]*

S2. Prove Proposition 26 from the notes: Let U, W be subspaces of a finite-dimensional vector space V. The following are equivalent:

(i) $V = U \oplus W$;

(ii) every $v \in V$ has a unique expression as $u + w$ where $u \in U$ and $w \in W$;

(iii) $\dim V = \dim U + \dim W$ and $V = U + W$;

(iv) $\dim V = \dim U + \dim W$ and $U \cap W = \{0_V\}$;

(v) if $u_1, \ldots, u_m$ is a basis for U and $w_1, \ldots, w_n$ is a basis for W, then $u_1, \ldots, u_m, w_1, \ldots, w_n$ is a basis for V.

Since proving (i) $\Leftrightarrow$ (ii) is Q6(a) on this sheet, I'm not going to write out a solution here.

We could go about proving all possible other implications, but that is more work than is necessary: we can strategically just prove a few.

(i) $\Rightarrow$ (iii): Assume that $V = U \oplus W$. Then $U \cap W = \{0\}$ so $\dim(U \cap W) = 0$. Also, $V = U + W$. Hence, by the dimension formula, $\dim V = \dim(U + W) = \dim U + \dim W - \dim(U \cap W) = \dim U + \dim W$.

(iii) $\Rightarrow$ (iv): Assume that $\dim V = \dim U + \dim W$ and $V = U + W$. Then $\dim(U + W) = \dim U + \dim W$, so by the dimension formula $\dim(U \cap W) = 0$, so $U \cap W = \{0\}$.

(iv) $\Rightarrow$ (i): Assume that $\dim V = \dim U + \dim W$ and $U \cap W = \{0\}$. Then, by the dimension formula, $\dim(U + W) = \dim U + \dim W - \dim(U \cap W) = \dim U + \dim W = \dim V$. So $U + W$ is a subspace of V with the same dimension as V, so $U + W = V$. But since by assumption we also have $U \cap W = \{0\}$, this gives $V = U \oplus W$.

[So now we know that (i), (iii) and (iv) are all equivalent.]

(i) $\Rightarrow$ (v): Assume that $V = U \oplus W$. Take $u_1, \ldots, u_m$ a basis for U and $w_1, \ldots, w_n$ a basis for W. Let $S = \{u_1, \ldots, u_m, w_1, \ldots, w_n\}$.

S spans V: Take $v \in V$. Then $v = u + w$ for some $u \in U$ and $w \in W$. Now $u = \alpha_1 u_1 + \cdots + \alpha_m u_m$ and $w = \beta_1 w_1 + \cdots + \beta_n w_n$ for some α_i, $\beta_j \in \mathbb{F}$, so v is a linear combination of elements of S.

S linear independent: Take α_i, $\beta_j \in \mathbb{F}$ with $\alpha_1 u_1 + \cdots + \alpha_m u_m + \beta_1 w_1 + \cdots + \beta_n w_n = 0$. Then $\alpha_1 u_1 + \cdots + \alpha_m u_m = -\beta_1 w_1 - \cdots - \beta_n w_n \in U \cap W$, so both sides are 0. But $u_1, \ldots, u_m$ are linearly independent, so $\alpha_1 = \cdots = \alpha_m = 0$, and similarly for β_j.

So S is a basis for V.

(v) $\Rightarrow$ (iii): Assume that if $u_1, \ldots, u_m$ is a basis for U and $w_1, \ldots, w_n$ is a basis for W, then $u_1, \ldots, u_m, w_1, \ldots, w_n$ is a basis for V.

Then we see immediately that $\dim V = \dim U + \dim W$, and that $V = U + W$.

S3. Find an example to show that it is not the case that if $V = U \oplus W$ then every basis of V is a union of a basis of U and a basis of W.

Here is one example, of course there are many possible examples that you might choose.

Let $V = \mathbb{R}^2$, let $U = \{(x\ 0) : x \in \mathbb{R}\}$, let $W = \{(0\ y) : y \in \mathbb{R}\}$. Then $V = U \oplus W$.

Consider the basis $\{(1\ -1), (2\ 3)\}$ of V — this is indeed a basis, because it is a set of two linearly independent elements in a vector space of dimension 2. Then neither basis vector is in U, and neither is in W, so neither can be a basis vector for U or for W. So this basis of V is not a union of a basis of U and a basis of W.